SYMMETRY-PROTECTED TOPOLOGICAL PHASES AND QUANTUM ANOMALIES

BY

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DISSERTATION

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Abstract

We present the correspondence between symmetry-protected topological (SPT) phases and their anomalous boundary states, based on examples in various spacetime dimensions. Through the study of the effect of interactions on these SPT phases, we discovered a new formalism of quantum anomalies, associated with discrete spacetime (such as time-reversal and spatial reflection) symmetries in particular, to classify distinct interacting topological phases. An example is the $\mathbb{Z}_2$ classification of the (2+1)d topological insulator protected by charge U(1) and time-reversal (or CP) symmetries, which can be deduced by the form of the global U(1) gauge anomalies on its edge theories defined on closed unorientable manifolds. In this case, the nontrivial phase (in free systems) is robust against electron interactions. Another example is the (3+1)d topological superconductor protected by only time-reversal or reflection symmetry. For this system, we identified the bulk phase by studying the global gravitational anomalies of the surface theories formulated on unorientable spacetime manifolds, and also discussed its connection to the collapse of the non-interacting classification by an integer $\mathbb{Z}$ to $\mathbb{Z}_{16}$, in the presence of interactions.

We also revisit the problem of gauging a discrete internal symmetry in theories of chiral (Weyl) fermions in 3+1 dimensions – which have not been fully understood so far – from the perspective of fermionic SPT phases in 4+1 dimensions. Comparing with the previous results, we give a complete answer for the anomalies constraints on the discrete symmetry, as our approach is based on purely geometrical considerations, namely, our assumption is more fundamental and general. Furthermore, our result also provides an understanding of gapped states of fermions with anomalous discrete symmetries, and we present a model, based on weak coupling, for constructing these anomalous gapped states.

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Chapter 1

Introduction

1.1 Background and motivation

Topological phases are gapped phases of matter that can not be deformed to a trivial phases, such as an atomic insulator, without the occurrence of quantum phases transition. Topologically distinct phases are always separated by intervening gapless phases or quantum critical points. A classic example of a topological phase in two spatial dimensions is the integer quantum Hall effect (IQHE). Quite often, it is meaningful to discuss the "theory space" – in the language of the renormalization group – in the presence of symmetries. Under symmetry consideration, a new topological distinction among quantum phases may be created. Symmetry-protected topological (SPT) phases of matter are defined in this way – they are topologically equivalent to trivial states of matter in the absence of symmetries, while once a certain set of symmetries are imposed, they are adiabatically disconnected to trivial phases. Canonical examples of SPT phases include the Haldane phase realized in one-dimensional quantum spin chains [1], the quantum spin Hall effect (QSHE) in two dimensions [2, 3, 4], and the three-dimensional time-reversal symmetric topological insulator [5]. For partial references for recent works on symmetry protected topological phases, see Refs. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

While SPT phases do not have intrinsic topological orders, namely, they do not support (deconfined) fractional excitations, they are sharply (topologically) distinct from topologically trivial states. In other words, the distinction between SPT and trivial phases cannot be made within Landau's theory, so SPT phases are beyond the classification of phases of matter based on broken symmetries. By definition, when going from an SPT phase to a trivial phase in a phase diagram by changing parameters in the system's Hamiltonian, one inevitably encounters a quantum phase transition, if the symmetry conditions are strictly enforced. This in turn implies that if an SPT phase is spatially proximate to a trivial phase, there should be a gapless state localized at the boundary between the two phases; this critical state can be thought of as a "phase transition" occurring locally in space, instead of the parameter space of the Hamiltonian. As implied by this construction, the edge state of a non-trivial SPT phase should never be removable (completely gapped) if the symmetries are strictly imposed. (It should be noted, however, that there are other interesting possibilities for symmetric surface states for (3+1)d SPT phases [20, 21, 22, 23].) Hence, this critical boundary state signals the topological distinction between the SPT and trivial phases, and many properties of SPT phases can be extracted from their boundary physics. For example, by inspecting under which symmetry conditions a given edge theory is stable/unstable, one can predict under which symmetry conditions a given phase can be an SPT phase.

Although studying the boundary instead of the bulk reduces the dimensionality of the problem, it is still not straightforward to judge if a given state is topological or not. In principle, one could enumerate all possible symmetry-allowed perturbations within the edge theory, which can potentially gap out the edge. Without any guiding principle, however, such "brute force" approach is quite cumbersome, and also, more fundamentally, does not provide any intuition on the physics of SPT phases. Hence it is necessary to have an efficient and illuminating guiding principle for diagnosing topological properties of phases of matter with symmetries.

While non-interacting topological phases in systems of electrons are considerably well studied – as a fairly general topological classification of non-interacting fermion systems is possible [9, 10, 11], an important next challenge is to understand the interaction effects on these exotic phases of matter. In thinking about the interplay of interactions, symmetries, and topological phenomena in generic gapped systems, some questions naturally present themselves. One is about the stability of free (fermionic) SPT phases, such as topological band insulators and fully gapped superconductors, against many-body interactions. In fact, there are known examples where a would-be SPT phase, for which one can define a topological invariant of some sort at the level of single-particle wave functions, can actually be adiabatically connected to a topologically trivial phase once one includes interactions [24, 25, 26, 27, 28, 29]. Another is the study of interaction-driven SPT phases — new kinds of topological phases emerging solely due to strong electron interactions and symmetry protection and might not have a free fermion counterpart. Examples include nontrivial bosonic topological phases arising in strongly interacting electronic systems protected by charge U(1) and time-reversal symmetries in three spatial dimensions [30].

In this thesis, we would focus on the first question, that is, to understand the underlying mechanism behind the collapse of noninteracting classification of some fermionic SPT phases. The main goal in this thesis is to provide a general scheme that allows us to judge if a given phase is an SPT phase or not, or, to be more precise, to diagnose under which symmetry condition a given phase can possibly be an SPT phase.

1.2 Methods and Main results

One of the most efficient and powerful methods to study SPT phases is to *twist* or *gauge* the underlying symmetries that protect SPT phases [28, 31, 32]. It was proposed that the twisted theory – or the "weakly gauged" theory that is coupled a background gauge field associated to the symmetries – can be used to diagnose the original SPT phases, that is, to judge whether or not the original theory is symmetry-protected and distinct from topologically trivial phases. More specifically, once twisted, the edge theory of an SPT phase suffers from various kinds of *quantum anomalies* – an intricate form of symmetry breaking caused by quantum effects. Already in Laughlin's gauge argument, a topological charge pumping process, that is, the non-invariance of the system's ground state under large U(1) gauge transformations, was used to establish the stability of the quantum Hall states against interactions and disorder [33]. For SPT phases, see recent works in Refs. [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44]. On the other hand, gauging (non-spatial) symmetries effectively deconfines a set of quasiparticles (anyons). The fractional statistics of the braiding in the gauged theory can be used to diagnose the original SPT phases [31, 45].

Since anomalies are known to be insensitive to whether the underlying fermions are interacting or not, they can be used to diagnose the effect of interaction on the non-interacting classification of fermionic SPT phases, and thus can provide a sharp definition of interacting topological phases. On the other hand, owing to the fact that the underlying fermion systems are topological, the field theories for the responses turn out to be described by anomalies. These anomalies describe the responses both, in the bulk and at the surface. The charge, spin, and thermal surface responses are examples.

The non-interacting SPT phases can be classified by studying the problem of Anderson localization at the sample boundaries or K-theory for the bulk theories – the so-called "ten-fold" classification scheme of topological insulators and superconductors in general dimensions [9, 10, 11]. On the other hand, the authors in [37] have shown the connection between anomalies and some topological phases with integer classification. These topological phases are robust again interactions, as their boundary support chiral gapless (critical) modes, which suffer perturbative anomalies, the anomalies associated with gauge or gravitational transformations that can be connected continuously to identity. An example is the (2+1)d integer quantum Hall insulator (class A). The (1+1)d chiral edge states (of the sample boundary) can have an U(1) gauge anomaly. The presence of this anomaly, which means the breakdown of charge conservation at the boundary, indicates that the current "leaks" into the bulk, which is nothing but the quantum Hall effect in the condensed matter setting. Actually, the U(1) gauge anomaly exists in all even space-time dimensions $D = 2k$, and such kind of anomaly predicts the presence of topological insulators in symmetry class A in $(2k+1)$ dimensions. In addition to gauge anomaly, there are other two kinds of perturbative anomalies: the purely gravitational anomaly that exists in $D = 4k+2$ and the mixed gauge-gravitational anomaly that exists in $D = 4k$. The presence of the anomaly in the formal case predicts the existence of topological superconductors in $D = 4k+3$ (e.g. class D in (2+1)d and class class C in (6+1)d [11]), while the anomaly in the latter case indicates the occurrence of topological insulators in $D = 4k+1$ (e.g. class AII in (4+1)d and class class AI in (8+1)d [11]).

In the absence of the (boundary) perturbative anomalies that exist on those space-time mentioned above, there might still exit nontrivial (bulk) topological phases in one higher dimensions. For these topological phases, characterized by either an integer or a $\mathbb{Z}_2$ invariant in the non-interacting classification, we would like to know:

(i) *Is the classification of these topological phases robust against interactions?*

(ii) *What kind of anomalies (other than the perturbative anomalies mentioned above), if present on the boundaries of these phases, can be used to characterize the bulk topological phases?*

For (i), people have realized that for some symmetry classes the non-interacting classification would not hold upon the inclusion of interactions. Some examples have been studies in low dimensions ($d \leq 3$) [24, 25, 26, 46, 27, 28, 29, 47, 48, 49, 41, 50, 51, 52, 53]. People found, in these cases, the effect of interactions can lead to either the collapse of non-interacting classification or the occurrence of nontrivial phases that are trivial in the non-interacting limit. For (ii), on the other hand, people proposed to use global anomalies (in the absence of perturbative anomalies), which are anomalies associated with "large" gauge or coordinate transformations that can not be reached continuously from the identity, to characterize interacting topological phases [37]. Following this idea, we have recently studied several cases of interacting topological phases in two and three dimensions from the prospects of global anomalies [54, 55], which will be presented respectively in chapters 2 and 3 in this thesis.

On the other hand, as one can use anomalies to detect and characterize SPT phases, it is also possible, conversely, to study the problem of anomalies associated with a symmetries from the perspective of (bulk) SPT phases in one dimension higher. Actually, this provides a more precise definition of quantum anomalies, especially for those associated with discrete symmetries. An example is a refinement of the "parity" anomaly on unorientable three-manifolds, which was initially studied in our work [55] and then explored further in Edward Witten's recent works [56, 57]. Immediately we realized the problem of anomalies in fermion theories with discrete internal symmetries, as originally studied in some early works [58, 59], could also be analyzed in a similar way as the one of the reformulated "parity" anomaly. By computing the partition function on a generic five-dimensional spin manifold with an additional (symmetry) structure, we eventually have a complete understanding of the anomaly of a discrete symmetry, by the anomaly inflow argument, in four dimensions. The result will be presented in chapter 4. Therefore, our work not only solves a difficult odd problem in fundamental physics, but also presents a way to understand a topological classification of spin manifolds (with extra structures) – as a problem in mathematics – by a quantum-field-theory approach.

1.3 Thesis overview

The rest of the thesis is organized as follows:

In chapter 2, we generalize Laughlin's flux insertion argument, originally discussed in the context of the quantum Hall effect, to topological phases protected by non-on-site unitary symmetries, in particular by parity symmetry or parity symmetry combined with an on-site unitary symmetry. As a model, we discuss fermionic or bosonic systems in two spatial dimensions with CP symmetry, which are, by the CPT theorem, related to time-reversal symmetric topological insulators (e.g., the quantum spin Hall effect). In particular, we develop the stability/instability (or "gappability"/"ingappablity") criteria for non-chiral conformal field theories with parity symmetry that may emerge as an edge state of a symmetry-protected topological phase. A necessary ingredient, as it turns out, is to consider the edge conformal field theories on unoriented surfaces, such as the Klein bottle, which arises naturally from enforcing parity symmetry by a projection operation.

In chapter 3, we identify quantum anomalies in two kinds of (3+1)d fermionic symmetry protected topological phases: (i) topological insulators protected by CP (charge conjugation $\times$ reflection) and electromagnetic U(1) symmetries, and (ii) topological superconductors protected by reflection symmetry. For the first example, which is related to, by CPT-theorem, time-reversal symmetric topological insulators, we show that the CP-projected partition function of the surface theory is not invariant under large U(1) gauge transformations, but picks up an anomalous sign, signaling a $\mathbb{Z}_2$ topological classification. Similarly, for the second example, which is related to, by CPT-theorem, time-reversal symmetric topological superconductors, we discuss the invariance/non-invariance of the partition function of the surface theory, defined on the three-torus and its descendants generated by the orientifold projection, under large diffeomorphisms (coordinate transformations). The connection to the collapse of the non-interacting classification by an integer ($\mathbb{Z}$) to $\mathbb{Z}_{16}$, in the presence of interactions, is discussed.

In chapter 4, we revisit the problem of gauging a discrete internal symmetry – $\mathbb{Z}_n$ symmetry in particular –

in theories of chiral (Weyl) fermions in 3+1 dimensions. Our approach is based on geometrical considerations, which are from the perspective of fermionic SPT phases in 4+1 dimensions. The anomaly constraints on the discrete gauge symmetry are derived by looking at the consistency of formulating the fermion theory on any four-dimensional spin manifold with a background gauge field associated to this symmetry. Our result agrees with the previous work by Ibanez and Ross [60]; however, our assumption is more fundamental, as the conditions we derive are independent of information about any high energy theory with continuous gauge symmetries in which a given $\mathbb{Z}_n$ gauge symmetry is embedded. We also give a classification of fermion theories with anomalous $\mathbb{Z}_n$ symmetries in 3+1 dimensions, which is represented by an Abelian group in terms of $\mathbb{Z}_n$ charges. Our result also provides a fundamental understanding of gapped states of fermions with anomalous discrete symmetries, and we present a model, based on weak coupling, for constructing these anomalous gapped states.

In chapter 5, we conclude and give future prospect of the results presented in this thesis.

Chapter 2

Symmetry-protected topological phases in 2+1 dimensions: A generalized Laughlin argument and orientifolds

2.1 Introduction

One of the most fundamental and defining properties of the quantum Hall effect (QHE) is the charge pumping discussed in Laughlin's thought experiment [33, 61]. This non-perturbative argument explains the extreme robustness of the QHE against disorder and interactions. In the language of quantum field theories, the charge pumping in Laughlin's gauge argument is a manifestation of a *quantum anomaly*, *i.e.*, the breakdown of a classical symmetry caused by quantum effects. This is an extreme case where quantum mechanical effects completely betray our expectations from classical physics. To be more precise, the quantum Hall state supports, in the presence of a boundary (an edge), a chiral edge state. If we focus on an edge, the total charge is not conserved within the edge, *i.e.*, the U(1) symmetry associated with the particle number conservation is violated, as the charge leaks into the bulk precisely because of the QHE. This well-known *bulk-boundary correspondence* of the QHE relates the bulk topological properties and the gauge anomaly (non-conservation of charge) at the edge.

The connection to a quantum anomaly gives the conceptual backbone of the QHE. In fact, it is desirable to connect topological phases of *any kind*, not just the QHE, to a quantum anomaly for the following reasons.

This chapter was written based on the result of a previous publication [54] of the dissertation author and collaborators.

First, quantum anomalies often provide a way to detect a non-trivial topological nature of the state (*e.g.*, charge pumping in the QHE described above), and hence gives an operational definition of a topological phase. Second, once a topological phase is characterized in terms of a quantum anomaly, it is most likely to be stable against interactions and disorder.

In this chapter, we will focus on SPT phases beyond the QHE in (2+1) dimensions, and discuss "gappability" of their (1+1)-dimensional [(1+1)D] edge states in the presence of symmetry conditions. We consider a given (1+1)D gapless (conformal) field theory, which may emerge as an edge state of a bulk theory, and ask if its gapless nature can be protected by some symmetry conditions. Once the ingappability of the edge state is established, the corresponding bulk theory cannot adiabatically be connected to a topologically trivial state that do not support a gapless edge theory – the state in question is in a SPT phase protected by the symmetries. On the other hand, if the edge theory turns out to be gappable in the presence of the symmetries, the bulk theory may be deformable to a topologically trivial state.

To diagnose gappability/ingappability of an edge theory, a generalization of Laughlin's gauge argument was proposed in Ref. [28], in a way that can be applied to edge states of SPT phases. The purpose of this paper is to extend the scheme proposed in Ref. [28] to study topological phases protected by unitary *non-on-site* symmetries, *e.g.*, parity symmetries. (We will mainly be interested in unitary symmetries, but *anti-unitary* symmetries such as time-reversal symmetry are also relevant to our discussion.)

A key ingredient of the strategy suggested in Ref. [28] is the strict enforcement of symmetry conditions by a projection operation, or, to be more precise, an "orbifolding" procedure in the edge conformal field theory (CFT). An orbifold of a theory, which is invariant under a global unitary on-site symmetry, is given by averaging the partition function over boundary conditions twisted by a group element in the symmetry group. Roughly speaking, this procedure removes states that are not invariant under the symmetry group. One can then study an adiabatic evolution of the projected ("orbifolded") partition function. For example, if a U(1) symmetry is conserved, as in Laughlin's original argument, one can ask if the orbifolded partition function is invariant or not under a large U(1) gauge transformation. While an original non-projected (non-orbifolded) theory may be anomaly free, once orbifolded, the edge theory may fail to perform an anomaly-free adiabatic process. Non-invariance of the orbifolded edge theory under a large U(1) gauge transformation signals non-trivial topological properties of the corresponding bulk state. One can also ask, perhaps more fundamentally, the invariance/non-invariance of the orbifolded system under large coordinate transformations, such as modular transformations on a space-time manifold with non-zero genus, *e.g.*, a torus.

This scheme is demonstrated to work for various examples [62, 63]. A similar projection or "gauging" procedure is also employed in Ref. [19], where a criterion for the ingappability/gappability of an edge theory is derived using fractional statistics of the defects obtained from gauging some unitary on-site global symmetry in a gapped bulk theory.

We extend the scheme proposed in Ref. [28] to SPT phases that are protected by unitary spatial symmetries, in particular to parity symmetry (denoted by $\mathscr{P}$ in the following). An interplay between spatial (and, more generally, crystal) symmetries and topological properties of electronic states have been studied extensively recently [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77]. Following the general strategy described above, we consider the orbifolding or gauging procedure by a parity symmetry. Unlike orbifolding a unitary on-site symmetry, orbifolding parity symmetry naturally leads to a change of the topology of the spacetime manifold of the edge theory. Once orbifolding parity symmetry, an edge theory is defined on an *unoriented* (1+1)D spacetime surface, such as the Klein bottle instead of a spacetime torus [78, 79, 80] We refer to these conformal field theories as *orientifold* [81, 82] conformal field theories [1].

In the presence of yet another symmetry (represented by a symmetry group $\mathscr{G}$ – the total symmetry group is $\mathscr{P} \rtimes \mathscr{G}$ or $\mathscr{P} \times \mathscr{G}$ in addition to parity, there is a simple consequence of the topology change from the torus to the Klein bottle, which can be inferred by comparing their fundamental groups, *i.e.*, the space of non-contractible loops on these surfaces. On the torus, there are two independent cycles and one can assign a group element to each cycle, *e.g.*, a U(1) phase factor; these two group elements (g_1 and g_2, say) represent boundary conditions along each cycle. On the Klein bottle, on the other hand, because of its unoriented nature, there is a certain restriction on the group elements that one can assign for cycles; while one of the group elements, g_1, say, can be any element in $\mathscr{G}$, the other group element g_2 must satisfy $g_2 = g_2^{-1}$ (see discussion near eq. (2.30) below for more details).

In this work, we focus on the cases where $\mathscr{G}$ is a U(1) symmetry (either charge U(1) or “spin” U(1) symmetry). One of our main observations is a crucial role played by the $\mathbb{Z}_2$ flux satisfying $(g_2)^2 = 1$, *i.e.*, along this cycle, the only allowed boundary conditions are periodic or antiperiodic. While g_1 can be used as an adiabatic parameter that we can use as a “knob” to implement a Laughlin argument (*i.e.*, an adiabatic evolution of the partition function as we change $g_1 \in \mathscr{G}$), g_2 turns out to be fixed by the type of parity symmetry $\mathscr{P}$. We will show in this paper that these distinctions by g_2-flux are closely related to the topological classification of parity symmetric systems.

While our methodology is applicable to a wider class of systems with parity symmetry, in this paper, we choose to work with systems with parity combined with charge conjugation symmetry (CP symmetry) [85]. Specifically, we consider three examples: (i) fermionic systems with conserved charge U(1) symmetry

[1] For a connection between orientifolds and time-reversal symmetric topological insulators and superconductors from the spacetime point of view (as opposed to the worldsheet point of view presented in this paper), see Refs. [83, 84].

and CP symmetry (CP symmetric topological insulators) [2]; (ii) bosonic systems with conserved charge U(1) and CP symmetries; and (iii) K-matrix theory with CP and U(1) symmetries. We discuss the topological classification of these systems by using the method of the generalized Laughlin argument.

The systems with CP symmetry are our canonical examples in the sense that they are closely related to SPT phases protected by time-reversal symmetry through the CPT theorem. For systems with Lorentz invariance, the CPT theorem tells us that any perturbation (mass terms and interactions) prohibited by T (CP) symmetry is also excluded by CP (T) symmetry. Thus, for Lorentz invariant systems, SPTs protected by time-reversal are automatically protected by CP symmetry as well.

For condensed matter systems, Lorentz invariance is not a prerequisite. However, for non-interacting fermions, it is known that the general topological classification can be obtained solely from the topological classification of Lorentz invariant Dirac Hamiltonians. When available, topological field theory descriptions of topological phases are also Lorentz invariant. In addition, it is known that, if one considers *the entanglement spectrum* as a tool to study SPT phases, CP symmetry of a physical Hamiltonian is translated to an effective time-reversal symmetry of the corresponding entanglement Hamiltonian if one bipartites the system into two subsystems that are related by CP symmetry [64, 85, 86]. For these reasons, our method also provides a new insight into time-reversal symmetric topological systems, including the QSHE, by relating them to orientifold conformal field theories. Thus we provide a method for "twisting" or "gauging" time-reversal symmetry. (See recent discussion in Refs. [87, 88].)

The main results and outline of this chapter can be summarized as follows: for the remainder of this section, we will review gauge and chiral anomalies in (1+1) dimensions and their connection to topological phases in (2+1) dimensions. In particular, we rephrase the original Laughlin argument in a quantum field theory language, which we will use for our later discussion.

In Sec. 2.2, we begin our discussion by introducing a free fermion model with CP and electromagnetic U(1) symmetries. We consider two kinds of CP symmetries, one which protects gapless edge states and hence leads to a non-trivial bulk symmetry-protected topological phase, and the other which does not lead to a topological insulator. An anomaly ("CP anomaly") is identified in the non-chiral edge states of the CP-symmetric topological insulator. We then present, by using the CP-symmetric topological insulator as an example, a generalization of Laughlin's argument to systems with parity symmetry (CP symmetry, in this case). By considering the partition function of the non-chiral edge theory with CP projection, it is shown that the distinction between the two cases shows up as the presence/absence of a $\mathbb{Z}_2$ flux on the Klein bottle ("g_2" in the above notation). Under an adiabatic insertion of electromagnetic U(1) flux ("g_1"

[2] The fermionic models with CP and charge U(1) symmetries can also be interpreted/realized as a BdG system with parity and spin U(1) symmeries (*e.g.*, z-component of spin, S_z, is conserved).

in the above notation), the projected partition function is anomalous/anomaly-free when the edge theory is ingappable/gappable.

In Sec. 2.3, the generalized Laughlin argument is applied to bosonic SPT phases with a single-component non-chiral boson edge theory with CP symmetry. The results are consistent with microscopic stability analysis of CP symmetric edge theories given in Ref. [85].

In Sec. 2.4, we consider a broader range of edge theories described by the K-matrix theory with CP symmetry. With the generalized Laughlin argument, we derive the stability criterion for the edge theories, which agrees with the stability criterion of the K-matrix theory with time-reversal symmetry [89, 90, 91], as expected from the CPT theorem.

We conclude in Sec. 2.5. In Appendix A.1, we discuss the eigenvalue of the CP transformation of the ground state of edge CFTs, and in particular its evolution under an adiabatic evolution of the background flux. Once we choose to preserve the U(1) symmetry, the CP eigenvalue must be independent of the background flux, which we assume for the bulk of the paper. On the other hand, an alternative point of view is possible where we strictly enforce the U(1) symmetry. Once this is done, CP symmetry may be anomalous, and hence the ground state CP eigenvalue may be dependent on the background flux. This issue is discussed in Appendix A.1 by making use of the state-operator correspondence of CFTs. Appendix A.2 explains a technical detail that arises when diagnosing the stability of K-matrix edge theories in Sec. 2.4.

2.1.1 The integer QHE and gauge anomaly

For chiral topological phases in two spatial dimensions, their edge states (which are chiral) are anomalous. When there is the electromagnetic U(1) symmetry, the chiral edge states are anomalous under infinitesimal as well as large U(1) gauge transformations. Even in the absence of the electromagnetic U(1) invariance, the edge states are still anomalous under infinitesimal as well as large diffeomorphisms.

For later use, let us review the anomaly under large U(1) gauge transformations at the edge of the integer QHE. (See, for example, Refs. [92, 93, 94, 95], for discussion on the edge theory of various quantum Hall states). (We will follow notations in Ref. [96].) The chiral edge mode of the integer QHE is described by the action

$$S = \frac{1}{2\pi} \int dt dx \, i\psi_R^\dagger (\partial_t + \partial_x)\psi_R, \tag{2.1}$$

where (t, x) is the spacetime coordinate of the edge theory, and chirality is chosen, say, to arrive at the right-moving fermions.

Following Laughlin's thought experiment, we now insert magnetic flux into the system of a cylindrical shape. In terms of the fermion field in the edge theory, this amounts to imposing the following twisted boundary conditions both for space and time directions:

$$\psi_R(t, x + 2\pi) = e^{2\pi i a}\psi_R(t, x),$$
$$\psi_R(t + 2\pi\tau_2, x + 2\pi\tau_1) = e^{2\pi i b}\psi_R(t, x). \tag{2.2}$$

Here the edge theory is defined on a spatial circle of radius 2π, and $\tau = \tau_1 + i\tau_2$ is the modular parameter of the spacetime torus. Under these boundary conditions, the right-moving partition function is computed to be [97]

$$\begin{aligned} Z_{[a,b]}(\tau) &= q^{-\frac{1}{24}+\frac{1}{2}(a-1/2)^2} e^{-2\pi i (b-1/2)(a-1/2)} \times \prod_{n=0}^{\infty}\left[1 + e^{-2\pi i(b-1/2)} q^{n+a}\right] \times \prod_{n=-1}^{-\infty}\left[1 + e^{+2\pi i(b-1/2)} q^{-n-a}\right] \\ &= \frac{1}{\eta(\tau)} \vartheta \begin{bmatrix} a - 1/2 \\ -b + 1/2 \end{bmatrix} (0, \tau), \end{aligned} \tag{2.3}$$

where $q = \exp(2\pi i \tau)$ and the theta function with characteristics is defined by

$$\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\nu, \tau) \equiv \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n+\alpha)^2} e^{2\pi i (\nu+\beta)(n+\alpha)}. \tag{2.4}$$

While the classical theory, defined in terms of the action (2.1) together with the boundary condition (2.2), is invariant under large gauge transformations $a \to a + 1$ and $b \to b + 1$, the partition function violates this invariance:

$$Z_{[a,b]} = Z_{[a+1,b]} = -e^{2\pi i a} Z_{[a,b+1]} = Z_{[-a,1-b]}, \tag{2.5}$$

and thus the edge theory is anomalous under this transformation [3].

2.1.2 The S_z conserving QSHE and chiral anomaly

As yet another exercise, let us consider a bulk topological insulator characterized by non-zero *spin* Chern-number. We require both charge U(1) and spin U(1) symmetries. The edge state, if it exists, also respects

[3] In the above calculations, the invariance is violated only for the temporal boundary condition. One in fact has a choice: by redefining $Z_{[a,b]} \to Z_{[a,b]} e^{-2\pi i a b}$, the partition function is now anomalous for spatial boundary condition. Such multiplication of the phase is related to (re-) assignment the U(1) charge to the ground state. See later discussion for more details.

these symmetries, at least classically. However, either one of these U(1) symmetries must be spoiled by quantum mechanical effects. Let us for now insist on the conservation of electromagnetic U(1) charge. One can then introduce an electromagnetic vector potential A. We then consider a non-chiral fermion coupled with the electromagnetic U(1) gauge field,

$$S = \int d^2 z \left(i\psi_R^\dagger D_z \psi_R + i\psi_L^\dagger D_{\bar{z}} \psi_L \right), \tag{2.6}$$

where D_z is a covariant derivative with the electromagnetic U(1) gauge field A. As is well known, the theory is not invariant under chiral gauge transformations, which in the present context are gauge transformations associated to the spin U(1) symmetry. This has to do with the presence of a non-trivial bulk topological phase protected by charge U(1) and spin U(1) symmetries.

The chiral anomaly comes about since the path integral measure is not invariant under chiral transformations. Let us consider the case where the Chern number associated to the vector potential A is non-zero:

$$Ch = \frac{i}{2\pi} \int \mathrm{tr}\, F > 0, \tag{2.7}$$

where F is the field strength of the external U(1) gauge potential A. Then, by the index theorem, the number of ψ_L zero modes (= the number of $\psi_R^\dagger$ zero modes) is larger by Ch than the number of $\psi_L^\dagger$ zero modes (= the number of ψ_R zero modes). The path integral measure is given by

$$\mathcal{D}\left[\psi^\dagger, \psi\right] = \prod_{\alpha=1}^{Ch} da_\alpha da_\alpha^* \prod_{n=1}^{\infty} db_n dc_n db_n^* dc_n^*, \tag{2.8}$$

where $\prod_{n=1}^{\infty} db_n dc_n db_n^* dc_n^*$ represents the "oscillator" part of the measure, whereas $\prod_{\alpha=1}^{Ch} da_\alpha da_\alpha^*$ represents the measure associated to the zero modes. While the measure $db_n dc_n db_n^* dc_n^*$ is invariant under both electromagnetic and spin U(1) global transformations, the zero mode part $da_\alpha da_\alpha^*$ has electromagnetic (vector) charge zero, but axial charge 2. Thus, the path integral measure is not invariant under the axial (spin) U(1) rotation. In fact, in the presence of nonzero flux with the Chern number Ch, the axial U(1) is broken down to its $\mathbb{Z}_{2Ch}$ subgroup.

To summarize, once we demand the electromagnetic U(1) symmetry to be preserved then, the chiral anomaly tells us that the spin U(1) [axial U(1)] must be broken at the edge – this is nothing but the QSHE, *i.e.*, the spin quantum number is pumped by an adiabatic threading of the electromagnetic flux.

In fact, one has a choice – if one decides to preserve spin U(1) symmetry, instead of charge U(1), one could thread "spin flux" and consider the corresponding spin vector potential. Going through the above argument, one then concludes the charge is not conserved. This has to do with charge pumping by insertion of spin flux.

2.2 2D fermionic topological phases protected by CP symmetry

In this section, we describe our methodology (a generalization of Laughlin's argument) in terms of a simple two-dimensional fermionic system (although the method applies to a wider class of systems). The system of interest conserves the electromagnetic U(1) charge and respects a discrete symmetry, CP, that is a combination of parity, P: $(x, y) \to (-x, y)$ in two spatial dimensions, and charge conjugation, C, which is a unitary Z_2 on-site symmetry.

By the CPT theorem, the CP symmetric system (the CP symmetric topological insulator) is related to the time-reversal symmetric topological insulator. (In fact, they are equivalent when there is Lorentz invariance.) As two-dimensional insulators with time-reversal symmetry that squares to -1 are classified in terms of the Kane-Mele $\mathbb{Z}_2$ topological invariant, so are CP symmetric insulators. The CP symmetric fermionic system can also be interpreted as a topological superconductor that conserves parity and the z-component of spin (this is an example of "T-duality"). See Ref. [85] for more details of superconducting systems equivalent (dual) to CP symmetric insulators.

2.2.1 CP symmetric insulators

The bulk tight-binding model A lattice model of the topological insulator with CP symmetry can be constructed on the two-dimensional square lattice by taking two copies of the above two-band Chern insulator with opposite chiralities. Consider the Hamiltonian in momentum space,

$$H = \sum_{k \in \mathrm{BZ}} \Psi^\dagger(k) \mathcal{H}(k) \Psi(k), \tag{2.9}$$

where $\Psi(k)$ is a four-component fermion field with momentum k, BZ represents the first Brillouin zone of the two-dimensional square lattice, and the single particle Hamiltonian in momentum space is given in terms of the 4×4 matrix as,

$$\mathcal{H}(k) = n_x(k)\tau_z\sigma_x + n_y(k)\tau_0\sigma_y + n_z(k)\tau_0\sigma_z, \tag{2.10}$$

where σ_μ and τ_μ ($\mu = 0, 1, 2, 3$) are two sets of Pauli matrices with σ_0 and τ_0 being a 2×2 unit matrix. The k-dependent three-component vector is given by

$$\vec{n}(k) = \begin{bmatrix} -\sin k_x \\ -\sin k_y \\ (\cos k_x + \cos k_y) + \mu \end{bmatrix}. \tag{2.11}$$

We will focus on the region $-2 < \mu < 0$ or $0 < \mu < +2$.

The Hamiltonian is invariant under the following two CP transformations

$$(\mathcal{CP})\Psi(r)(\mathcal{CP})^{-1} = U_{\mathrm{CP}}\Psi^\dagger(\tilde{r}), \tag{2.12}$$

where $r = (x, y)$ labels sites on the square lattice, $\tilde{r} := (-x, y)$ and the 2×2 unitary matrix U_{CP} is given by either of

$$\begin{aligned} U_{\mathrm{CP}} &= \tau_x\sigma_x \quad U^T_{\mathrm{CP}} = +U_{\mathrm{CP}}, \quad (\eta = +1), \\ U_{\mathrm{CP}} &= \tau_y\sigma_x, \quad U^T_{\mathrm{CP}} = -U_{\mathrm{CP}}, \quad (\eta = -1). \end{aligned} \tag{2.13}$$

To distinguish these two cases, we have introduced an index η; $\eta = \pm 1$ refers to the first/second case. We will also use the notation $\eta = e^{2\pi i \epsilon}$ where $\epsilon = 0, 1/2$ for $\eta = 1, -1$, respectively. The distinction between these two CP symmetries can be summarized as

$$(\mathcal{CP})^2 = e^{2\pi i \epsilon N_f} \tag{2.14}$$

where CP acts on states with N_f fermions.

It turns out that imposing $U_{\mathrm{CP}} = \tau_x\sigma_x$ ($\eta = 1$) leads to CP symmetric topological insulators. This can be seen by looking at the stability of the edge mode that can appear when we terminate the system in the y-direction (*i.e.*, the edge is along the x-direction.) One can check, numerically, and also in terms of the continuum edge theory (see below), $U_{\mathrm{CP}} = \tau_x\sigma_x$ protects the edge state while $U_{\mathrm{CP}} = \tau_y\sigma_x$ does not in the presence of the electromagnetic U(1) symmetry.

The edge theory We now develop a continuum theory for the edge modes along the x-direction. The edge theory is described by, at low-energies, the free fermion Hamiltonian with relativistic dispersion:

$$H = \frac{v}{2\pi}\int dx\big(\psi_L^\dagger i\partial_x\psi_L - \psi_R^\dagger i\partial_x\psi_R\big), \tag{2.15}$$

where the single-component complex fermion field operators ψ_L and ψ_R represent left-moving and right-moving electrons, and v is the Fermi velocity. The Hamiltonian conserves the U(1) charge

$$F_V = \int dx\,\Big[\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L\Big]. \tag{2.16}$$

The subscript "V" here represents the fact that this is a vector U(1) charge, as opposed to an axial U(1) charge,

$$F_A = \int dx\,\Big[\psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L\Big]. \tag{2.17}$$

Since the edge runs along x-direction, the Hamiltonian with the edge preserves (is consistent with) CP symmetry, *i.e.*, CP transformation is closed within the edge. Corresponding to the bulk CP transformations, we consider the following two types of CP symmetry operations that act within the edge theory [4]

$$\begin{aligned}(\mathcal{CP})\psi_L(x)(\mathcal{CP})^{-1} &= \psi_R^\dagger(-x),\\ (\mathcal{CP})\psi_R(x)(\mathcal{CP})^{-1} &= \eta\psi_L^\dagger(-x).\end{aligned} \tag{2.18}$$

As in the bulk, the sign $\eta = \pm$ distinguishes the cases of $(\mathcal{CP})^2 = 1$ and $(\mathcal{CP})^2 = (-1)^{N_f}$, respectively. They correspond to topological/non-topological cases; there are two uniform fermion mass bilinears that are consistent with the charge U(1) symmetry, $M_1 = \psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L$, and $M_2 = -i\big(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\big)$. These masses are odd under CP and prohibited when $\eta = +1$, whereas they are even under CP with $\eta = -1$. Thus,

$$\eta = e^{2\pi i\epsilon} = \begin{cases} +1 & \text{"topological"}\\ -1 & \text{"trivial"}.\end{cases} \tag{2.19}$$

[4] In the presence of both U(1) and CP symmetries, by combining these two symmetries, one can generate a series of CP transformations [98]:

$$\begin{aligned}\mathcal{U}_\alpha\psi_L(x)\mathcal{U}_\alpha^{-1} &= e^{i\alpha}\psi_R^\dagger(-x),\\ \mathcal{U}_\alpha\psi_R(x)\mathcal{U}_\alpha^{-1} &= \eta e^{i\alpha}\psi_L^\dagger(-x),\end{aligned}$$

where $\mathcal{U}_\alpha = e^{i\alpha F_V}\mathcal{CP}$. While these transformations are each qualified to be called CP, the inclusion of such a U(1) phase factor does not play any essential role in our discussion.

We conclude that the gapless edge theory is, at least at the *quadratic level*, stable (ingappable); the stability/instability of the edge theory in the presence of interactions is one of our main focuses in the following sections.

Let us consider, in addition, quadratic but spatially inhomogeneous perturbations. The two uniform mass terms M_1 and M_2 are not allowed in the presence of CP symmetry with $\epsilon = 0$. However, one can still consider $\int dx\, M(x)$ where, $M(x) = a_1(x)M_1 + a_2(x)M_2$, as a perturbation to the edge theory. The perturbation is not allowed by CP symmetry if $a_1(x)$ is constant but allowed if $\vec{a}(-x) = -\vec{a}(x)$. This perturbation gaps out most of the edge theory, but not completely. At the point $x = 0$ which is invariant under CP symmetry, it leaves a zero energy mode. This is a type of zero energy mode akin to the zero energy bound state in a soliton in polyacetyline, and carries 1/2 charge. This is also similar to the mass domain wall in the helical edge mode of the QSHE discussed previously [99]. The difference, however, is that for the time-reversal symmetric quantum spin Hall effect, the mass domain wall breaks TRS; the only exception being the location of the kink. In the CP symmetric case, the mass domain wall, as a whole, preserves CP symmetry.

2.2.2 CP anomaly

Although the edge theory cannot be gapped at the quadratic level, whether or not it is gappable under arbitrary symmetry-preserving perturbations is not clear; a state that appears to be non-deformable to a trivial state may actually be deformable to a trivial state once one includes perturbations beyond the quadratic level. For the CP symmetric topological phase defined above, we now try to develop a generalized Laughlin argument. To put it differently, we will ask if there is a quantum anomaly or not that may guarantee the gapless nature of the edge theory.

Quite often, non-chiral theories are anomaly-free, and hence are not qualified as a topological phase without symmetry condition. However, there may be a tension between symmetry conditions and an attempt to make the theory self-consistent (anomaly-free). To be more precise, if one insists on the symmetry conditions, one might not be able to achieve anomaly-freedom. In the scheme proposed in Ref. [28], the symmetry conditions are strictly enforced by the projection operations. Subsequently, we ask if the projected theory is anomaly-free or not.

We now show that there is indeed a tension between CP and charge conservation symmetries when $\epsilon = 0$. That is, if one enforces one of them, the other will be violated by quantum effects. In the following, as a warm-up, we first enforce charge conservation and then in this case see that CP will be violated. After that, we show that enforcing CP symmetry will violate the charge conservation. The latter can be thought of as a generalization of Laughlin's argument to symmetry-protected topological phases with charge U(1) symmetry.

The QSHE described in the introduction is an SPT phase protected by on-site $[\mathrm{U}(1) \times \mathrm{U}(1)]$ symmetry and characterized by an integer topological invariant. We now move on to a symmetry-protected topological insulator protected by non-on-site symmetry, and characterized by a $\mathbb{Z}_2$ topological invariant. Similar to the case of the S_z conserving QSHE, we can identify an anomaly in the edge theory. It is the parity anomaly discussed in Ref. [100].

The argument goes as follows: as in the case of the chiral anomaly, we consider a background field with non-zero Chern number Ch. When non-zero and positive, there are zero modes in ψ_L and $\psi_R^{\dagger}$, and the path-integral measure has a factor

$$\prod_{\alpha=1}^{Ch} da_{\alpha} da_{\alpha}^{*}. \tag{2.20}$$

Observe now that the Chern number Ch flips its sign under parity, P. It also flips its sign under the charge conjugation, C. Thus, the Chern number remains invariant under the combination of CP. Let us first consider the case of $\eta = +1$, where CP transformation is given by $(\mathcal{CP})\psi_L(x)(\mathcal{CP})^{-1} = \psi_R^{\dagger}(-x)$, $(\mathcal{CP})\psi_R(x)(\mathcal{CP})^{-1} = \psi_L^{\dagger}(-x)$. Thus, by CP, the $(\psi_L, \psi_R^{\dagger})$ zero modes in the background $A(x)$ are sent to the $(\psi_R^{\dagger}, \psi_L)$ zero modes in the background $-A(\tilde{x})$. Because of the Fermi statistics, the measure is transformed as

$$\prod_{\alpha=1}^{Ch} da_{\alpha} da_{\alpha}^{*} \to (-1)^{Ch} \prod_{\alpha=1}^{Ch} da_{\alpha} da_{\alpha}^{*}. \tag{2.21}$$

Since the field configurations $A(x)$ and $-A(\tilde{x})$ are smoothly connected, there is no way to define the measure so that it is invariant under CP. As we have seen, this case corresponds to topologically non-trivial bulk. On the other hand, when $\eta = -1$, $(\mathcal{CP})\psi_L(x)(\mathcal{CP})^{-1} = \psi_R^{\dagger}(-x)$, $(\mathcal{CP})\psi_R(x)(\mathcal{CP})^{-1} = -\psi_L^{\dagger}(-x)$, with the extra minus sign, the measure is invariant. In the next section, we make contact between these two cases (the case with and without parity anomaly) and topologically non-trivial and trivial insulators.

2.2.3 Generalized Laughlin's argument

In the above considerations, we have (implicitly) assumed that the electromagnetic charge U(1) is strictly conserved. However, it would be possible to instead demand CP symmetry to be strictly conserved. Given a conflict between CP symmetry and charge conservation (when $\epsilon = 0$) suggested by the above argument, it would not be possible to preserve electromagnetic U(1) charge symmetry once we demand CP symmetry.

This suggests the following: let us twist the boundary condition by the conserved electromagnetic U(1) charge (denoted by a and b as before in our discussion in the QH edge). The partition function depends on these twisting angles. One can then enforce CP symmetry by performing a projection on to a space with definite CP eigenvalue. In the path integral picture, the enforcement of CP symmetry leads to a conformal field theory defined on an unoriented spacetime, *i.e.*, the spacetime of the edge theory has the topology of the Klein bottle. Once we insist on CP symmetry, one may not be able to achieve gauge invariance under (large) U(1) gauge transformations. Equivalently, the partition function would not be invariant under $a \to a+1$ or $b \to b+1$. We view this conflict between the charge U(1) and CP symmetries as a signal for the existence of a bulk topological phase.

Twisted boundary conditions Let us now canonically quantize the fermion theory in the presence of the following spatial boundary condition:

$$\begin{aligned}
&\psi_L(x+\ell_1) = e^{2\pi i \nu_L}\psi_L(x), \\
&\psi_R(x+\ell_1) = e^{2\pi i \nu_R}\psi_R(x).
\end{aligned} \tag{2.22}$$

where the edge theory is put on a circle of circumference ℓ_1. A discrete symmetry (CP, in our example) may be compatible/incompatible with the boundary condition. By acting with CP on the boundary condition (2.22),

$$\begin{aligned}
&(\mathcal{CP})\psi_L(x+\ell_1)(\mathcal{CP})^{-1} = e^{2\pi i \nu_L}(\mathcal{CP})\psi_L(x)(\mathcal{CP})^{-1} \\
&\Rightarrow \psi^{\dagger}_R(-x-\ell_1) = e^{2\pi i \nu_L}\psi^{\dagger}_R(-x) \\
&\Rightarrow e^{2\pi i \nu_L}\psi_R(x-\ell_1) = \psi_R(x),
\end{aligned} \tag{2.23}$$

we conclude that CP symmetry is consistent with the twisted boundary condition when $\nu_L = \nu_R$, *i.e.*, only charge twist is allowed, $\psi_L(x+\ell_1) = e^{2\pi i \nu}\psi_L(x)$, $\psi_R(x+\ell_1) = e^{2\pi i \nu}\psi_R(x)$. By similar considerations, P symmetry is consistent with the twisted boundary condition only when $\nu_L = -\nu_R$ (*i.e.*, only the spin twist is allowed), and C symmetry is consistent with the twisted boundary condition only when $\nu_L = 0, 1/2$ and $\nu_R = 0, 1/2$.

The torus partition function For the CP symmetric case, we thus consider the spatial boundary condition with $\nu_L = \nu_R = a$. The corresponding partition function on the torus is

$$
\begin{aligned}
Z(\tau, \bar{\tau}) &= \mathrm{Tr}_{a\otimes a} \left[e^{-2\pi i(b-1/2)F_V} q^{L_R} \bar{q}^{L_L} \right] \\
&= Z_{[a,b]}(\tau) \overline{Z_{[a,b]}(\tau)},
\end{aligned} \tag{2.24}
$$

where $\overline{\cdots}$ denotes complex conjugation, and the Hamiltonian $H = H_R + H_L$ is given in terms of the left- and right-moving parts as

$$
\begin{aligned}
L_R &= L_0 - \frac{c_R}{24}, \quad L_L = \bar{L}_0 - \frac{c_L}{24}, \\
H_R &= \frac{2\pi v}{\ell_1} L_R, \quad H_L = \frac{2\pi v}{\ell_1} L_L,
\end{aligned} \tag{2.25}
$$

with $c_L = c_R = 1$. We have introduced the modular parameter through

$$
q = e^{2\pi i \tau}, \quad \tau = \tau_1 + i\tau_2, \quad \tau_2 = \frac{v\ell_2}{\ell_1}. \tag{2.26}
$$

Here, ℓ_2 represents the inverse temperature and we have included, in addition to the (imaginary) time translation generated by the Hamiltonian, the space translation generated by the momentum with the corresponding periodicity τ_1. (As we will see, τ_1 will not play any role once we impose CP symmetry.)

Written as a product $Z_{[a,b]}(\tau)\overline{Z_{[a,b]}(\tau)}$, where $Z_{[a,b]}(\tau)$ is given by Eq. (2.3), the partition function is large gauge invariant under $b \to b+1$ and $a \to a+1$. One can also check that the partition function is modular invariant.

The Klein bottle partition function with CP symmetry Let us now consider the partition function with CP projection:

$$
Z^{\mathrm{Proj}}_{[a]}(\tau) = \mathrm{Tr}_{a\otimes a} \left[\frac{1+\mathcal{CP}}{2} e^{-2\pi i(b-1/2)F_V} q^{L_R} \bar{q}^{L_L} \right] \tag{2.27}
$$

where we have inserted a projection operator, $(1+\mathcal{CP})/2$. The first term in the projection gives nothing but the torus partition function. The second term can be interpreted as a path integral over the fermion fields on the Klein bottle (with twisted boundary condition in the time direction.) To develop this picture, we first perform the Wick rotation $t = -ix_2$. The insertion of CP operator into the trace has the effect that by translating a fermion field ψ_R, say, once around the time direction, it comes back as $(\mathcal{CP})\psi_R(\mathcal{CP})^{-1}$. Thus, the time direction boundary condition is ($\ell_2 = 2\pi\tau_2$)

$$\begin{aligned}
\psi_R(x_1,x_2) &= -(\mathcal{CP})\psi_R(x_1,x_2+\ell_2)(\mathcal{CP})^{-1},\\
\psi_L(x_1,x_2) &= -(\mathcal{CP})\psi_L(x_1,x_2+\ell_2)(\mathcal{CP})^{-1},
\end{aligned} \tag{2.28}$$

where the factor -1 comes from the antiperiodic boundary condition of the fermion fields (we have set $b = 1/2$ for simplicity). *I.e.,*

$$\begin{aligned}
\psi_R(x_1,x_2) &= -\eta\psi_L^\dagger(-x_1,x_2+\ell_2),\\
\psi_L(x_1,x_2) &= -\psi_R^\dagger(-x_1,x_2+\ell_2).
\end{aligned} \tag{2.29}$$

(Observe that τ_1 is "projected out" by CP – see below.) The fermion fields are defined on the Klein bottle $(x_1,x_2) \equiv (x_1+\ell_1,x_2) \equiv (-x_1,x_2+\ell_2)$ with periodic boundary condition along x_1 (possibly twisted by a), but with the CP-twisted boundary condition along x_2 direction.

There is a simple consequence of the topology change from the torus to the Klein bottle induced by CP projection. The partition function on a Riemann surface of genus g (denoted by Σ_g) is given as the sum over all possible monodromies on Σ_g [101]:

$$Z^{\Sigma_g}(\tau) = \frac{1}{|\mathscr{G}|^g} \sum_{\alpha:\pi_1(\Sigma_g)\to\mathscr{G}} Z^{\Sigma_g}(\alpha;\tau), \tag{2.30}$$

= where τ are the moduli of Σ_g and $\pi_1(\Sigma_g)$ is the fundamental group of Σ_g. The group $\mathscr{G}$ is the symmetry group of the system which we make use of to implement Laughlin's flux threading argument (*i.e.,* to implement twisting boundary conditions on Σ_g). The total partition function $Z^{\Sigma_g}(\tau)$ is given as a sum over "all possible assignment" of group elements from $\mathscr{G}$ to non-contractible loops on Σ_g, and $Z^{\Sigma_g}(\alpha;\tau)$ denotes the partition function calculated with the particular set of monodromies α (*i.e.,* with particular set of "twisted" boundary conditions). For example, for the torus, the fundamental group is $\pi_1(T^2) = \langle \alpha,\beta | \alpha\beta\alpha^{-1}\beta^{-1} = 1\rangle$, *i.e.,* a group generated by two generators α and β representing two fundamental non-contractible loops) with a relation $\alpha\beta = \beta\alpha$. This means that, for any Abelian group $\mathscr{G}$, by considering a correspondence $\alpha \to g_1$, $\beta \to g_2$ where $g_{1,2} \in \mathscr{G}$, the summation is over $\mathrm{Hom}\,[\pi_1(T^2),\mathscr{G}] = (\mathscr{G})^2$. That is, the total partition function $Z^{T^2}(\tau)$ consists of partition functions $Z^{T^2}(\alpha;\tau)$, each of which represents a partition function with twisted boundary conditions ("fluxes") with group elements g_1 and g_2 along non-contractible loops α and β, respectively. On the other hand, for the Klein bottle, the fundamental group is given by $\pi_1(K) = \langle \alpha,\beta | \alpha\beta = \beta^{-1}\alpha\rangle$. For an Abelian group, this means $g_2 = (g_2)^{-1}$, *i.e.,* the flux or boundary condition is $\mathbb{Z}_2$ valued. As we will show below, the $\mathbb{Z}_2$ flux g_2 distinguishes topological and non-topological CP symmetric insulators.

Let us work out the effects of the projection explicitly. We first mode-expand the fermion fields as

$$\begin{aligned}\psi_R(x) &= \sqrt{\frac{2\pi}{\ell_1}} \sum_{r\in\mathbb{Z}+a} \psi_{R,r} e^{i\frac{2\pi}{\ell_1}rx}, \\ \psi_L(x) &= \sqrt{\frac{2\pi}{\ell_1}} \sum_{r\in\mathbb{Z}+a} \psi_{L,r} e^{i\frac{2\pi}{\ell_1}rx}.\end{aligned} \tag{2.31}$$

The CP transformation acts on the fermion modes as

$$\begin{aligned}(\mathcal{CP})\psi_{Lr}(\mathcal{CP})^{-1} &= \psi^\dagger_{Rr}, \\ (\mathcal{CP})\psi_{Rr}(\mathcal{CP})^{-1} &= \eta\psi^\dagger_{Lr}.\end{aligned} \tag{2.32}$$

For a given $r > 0$, there are four states, $|\mathrm{GS}_a\rangle$, $\psi^\dagger_{Rr}|\mathrm{GS}_a\rangle$, $\psi_{Lr}|\mathrm{GS}_a\rangle$, $\psi_{Lr}\psi^\dagger_{Rr}|\mathrm{GS}_a\rangle$, where $|\mathrm{GS}_a\rangle \propto \psi^\dagger_{Lr}|0\rangle$ is the ground state for the boundary condition specified by a. On these states, CP acts as, *e.g.*,

$$\begin{aligned}(\mathcal{CP})|\mathrm{GS}_a\rangle &= P_{[a]}|\mathrm{GS}_a\rangle, \\ (\mathcal{CP})\psi_{Lr}\psi^\dagger_{Rr}|\mathrm{GS}_a\rangle &= -\eta P_{[a]}\psi_{Lr}\psi^\dagger_{Rr}|\mathrm{GS}_a\rangle.\end{aligned} \tag{2.33}$$

Here, $P_{[a]}$, the CP eigenvalue of the ground state, is, *a priori*, undetermined. We have demanded that the system is CP invariant, and hence the first equation follows. Since CP is unitary, the eigenvalue $P_{[a]}$ should be a complex number of unit modulus.

For our discussion, it is crucial to know the a-dependence of the CP eigenvalue of the ground state. In particular, we need to compare the relative phase difference between $P_{[a]}$ and $P_{[a+1]}$. Under the assumption of the strict enforcement of CP symmetry, $P_{[a]}$ should be independent of a, and in particular, $P_{[a]} = P_{[a+1]}$. (If $P_{[a]}$ is dependent on a, the projection operation in fact is ill-defined.)

It is also insightful to use an alternative but equivalent picture for the effects of the fluxes a and b, where they are introduced as, instead of twisting angles for twisted boundary conditions, constant background gauge fields. In this picture, the Hamiltonian depends explicitly on a and is given by

$$H(a) = \frac{v}{2\pi}\int dx \big[\psi^\dagger_L i\partial_x\psi_L - \psi^\dagger_R i\partial_x\psi_R\big] + \frac{a}{\ell_1}J_V, \tag{2.34}$$

where $J_V = vF_A$ is the current operator. The fermion fields obey boundary conditions that are independent of a,

$$\psi_L(x+\ell_1) = \psi_L(x), \quad \psi_R(x+\ell_1) = \psi_R(x). \tag{2.35}$$

[5] Under an infinitesimal change in the flux $a \to a+\delta a$, since the perturbation commutes with CP and hence does not mix states with different eigenvalues of CP, the CP eigenvalue $P_{[a]}$ should be constant. For a similar discussion, see Refs. [102, 103, 104, 105, 106, 107, 108, 109] [6]. (See also discussion in Appendix A.1.)

Finally, the projected partition function can be calculated in a straightforward fashion, leading to

$$Z^{\mathrm{Klein}}_{[a]} = \mathrm{Tr}_{a\otimes a}\left[(\mathcal{CP})e^{-2\pi i(b-1/2)F_V} q^{L_R}\bar{q}^{L_L}\right] = \frac{P_{[a]}e^{2\pi i(a-1/2)(\epsilon-1/2)}}{\eta(q\bar{q})}\vartheta\left[\begin{array}{c} a-1/2 \\ -(\epsilon-1/2)\end{array}\right](0,q\bar{q}), \tag{2.36}$$

where we note $q\bar{q} = e^{2\pi i\tau}e^{-2\pi i\bar{\tau}} = e^{-4\pi\tau_2} = e^{-4\pi\frac{v\ell_2}{\ell_1}}$ and $\eta(q\bar{q}) = \eta(2i\mathrm{Im}\,\tau)$. Observe that the partition function is independent of τ_1, *i.e.,* it is projected out by CP. Similarly, the chemical potential b also does not show up in the projected partition function.

With $P_{[a]} = P_{[a+1]}$, which we enforce by CP symmetry, the partition function is invariant under $a \to a+1$ for the topologically trivial case ($\epsilon = 1/2$) whereas it is not for the topologically non-trivial case ($\epsilon = 0$),

$$Z^{\mathrm{Klein}}_{[a+1]} = e^{2\pi i(\epsilon-1/2)}Z^{\mathrm{Klein}}_{[a]}. \tag{2.37}$$

By comparison with the chiral partition function (2.3), we observe the distinction between topologically trivial ($\epsilon = 1/2$) and nontrivial ($\epsilon = 0$) cases shows up as a fictitious chemical potential (π flux in time direction). For the topological case the fermion effectively feels periodic boundary condition in time direction, whereas for the trivial case, the fermion effectively feels antiperiodic boundary condition. This anomaly vanishes if we consider two copies (more generally, an even number of copies) of the fermion theory, which suggests a $\mathbb{Z}_2$ classification of CP symmetric topological insulators [7].

[5] In addition to the Hamiltonian, there is a chemical potential which appears as an operator insertion $e^{-2\pi i(b-1/2)F_V}$ in the partition function. Viewing this operator as a part of the partition function, the system with the chemical potential is in general not invariant under CP since $(\mathcal{CP})F_V(\mathcal{CP})^{-1} = -F_V$. The only exceptions are the cases when $b = 0, 1/2$. When $b \neq 0, 1/2$, the system is not invariant under CP and so we cannot make a projection by CP symmetry. We therefore limit ourselves to $b = 0, 1/2$.

[6] The equivalence of the two pictures, one in terms of twisting boundary conditions, and the other in terms of background gauge fields, can be established by a gauge transformation that "unwinds" the boundary conditions, and vice versa. When the electromagnetic U(1) symmetry happens to be anomalous, care may be required in invoking such equivalence. (See, for example, Ref. [110].) In our approach, when an ambiguity such as the CP eigenvalue of the ground state arises, we follow what we expect in the absence of anomalies. We test the consistency of such an assumption arising from enforcement of CP symmetry with the electromagnetic U(1) symmetry by inspecting the behavior of the partition function under the adiabatic process of flux insertion.

[7] It is instructive to compare the CP projected partition function with the partition function with P projection. Parity

2.3 2D bosonic topological phases protected by CP symmetry

2.3.1 The edge theory

Armed with insights from the fermionic symmetry protected topological phases, we now discuss the bosonic topological phases. Below, we study the partition function of the edge of the bosonic CP symmetric topological insulator. We start from the single-component free boson theory on a ring of circumference ℓ defined by $Z = \int \mathcal{D}[\phi] \exp(iS)$ with the action

$$S = \frac{1}{4\pi\alpha'} \int dt \int_0^\ell dx \left[\frac{1}{v}(\partial_t \phi)^2 - v(\partial_x \phi)^2 \right], \tag{2.38}$$

where the ϕ-field is compactified with the compactification radius R as $\phi \equiv \phi + 2\pi R$; α' is the coupling constant of the boson theory. The canonical commutation relation is

$$[\phi(t,x), \partial_t \phi(t,x')] = 2\pi i \alpha' v \sum_{m \in \mathbb{Z}} \delta(x - x' - m\ell). \tag{2.39}$$

The theory can be quantized and decomposed into the left- and right-moving sectors. We introduce the chiral decomposition of the boson field ϕ as

$$\phi(t,x) = \varphi_L(x^+) + \varphi_R(x^-), \quad x^\pm := vt \pm x. \tag{2.40}$$

and also the dual boson field as

$$\theta(t,x) = \varphi_L(x^+) - \varphi_R(x^-). \tag{2.41}$$

---

transformation acts on fermion fields as

$$\begin{aligned} \mathcal{P}\psi_L(x)\mathcal{P}^{-1} &= \eta e^{i\alpha}\psi_R(-x), \\ \mathcal{P}\psi_R(x)\mathcal{P}^{-1} &= e^{i\alpha}\psi_L(-x). \end{aligned}$$

In our fermionic edge theory, by analyzing mass terms, one can check that there is no topological phase protected by parity symmetry (of any kind) and the electromagnetic U(1) symmetry. The absence of topological phases can be seen from the fact that the Klein bottle partition function with parity projection is anomaly-free. First recall that P symmetry is consistent with the twisting boundary condition only when $\nu_L = -\nu_R$. As we require only the U(1) charge conservation, this means only $\nu_L = \nu_R = 0$ (periodic boundary condition) or $\nu_L = -\nu_R = 1/2$ (antiperiodic boundary condition) are allowed. With this in mind, the projection works, for a given $r > 0$, as

$$\mathrm{Tr}\left[\mathcal{P}\, e^{-2\pi i(b-1/2)F_V} q^{H_R} \bar{q}^{H_L} \right] \propto \prod_r \left[1 + \eta e^{4\pi i(b-1/2)} e^{-2i\alpha} (q\bar{q})^r \right].$$

Thus, the phase α as well as η simply shifts the chemical potential b. In the case of P, we can freely change the time boundary condition b, but not the spatial boundary condition. Observe that this situation is opposite to what we had for CP symmetry. In the case of CP symmetry, we can freely change the space boundary condition, but not the time boundary condition. As before, we change $b \to b+1$ and ask if the theory is invariant under this large gauge transformation or not. Depending on the spatial boundary conditions, (periodic/antiperiodic), the partition function may pick up an anomalous phase. However, observe that the chemical potential enters in the partition function as $e^{4\pi i(b-1/2)}$ not $e^{2\pi i(b-1/2)}$. Due to this doubling, there are no anomalous phases.

As in the fermionic CP symmetric topological insulator, we consider two kinds of CP symmetries specified by $\epsilon = 0, 1/2$ as follows:

$$
\begin{aligned}
(\mathcal{CP})\phi(t,x)(\mathcal{CP})^{-1} &= -\phi(t,-x), \\
(\mathcal{CP})\theta(t,x)(\mathcal{CP})^{-1} &= +\theta(t,-x) + 2\pi\epsilon\alpha'/R.
\end{aligned}
\tag{2.42}
$$

This transformation law on the bosonic fields ϕ and θ under CP can be deduced from the transformation law of physical operators, the U(1) currents ($\partial_\mu\phi$ and $\partial_\mu\theta$) and vertex operators [$\exp i(\frac{k}{R}\phi + \frac{wR}{\alpha'}\theta)$ with $k, w \in \mathbb{Z}$] that describe local excitations, under CP. The single-component boson model with these CP symmetries is studied in Ref. [85], and it was demonstrated, based on microscopic analysis of gapping potentials, that the case with $\epsilon = 0$ is gappable while the case with $\epsilon = 1/2$ is not. We will reproduce this result from the generalized Laughlin argument with CP symmetry.

Quantization The mode expansions for the left- and right-moving boson fields is given by

$$
\begin{aligned}
\varphi_L(x^+) &= x_L + \pi\alpha' p_L \frac{x^+}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-\frac{2\pi i n x^+}{\ell}}, \\
\varphi_R(x^-) &= x_R + \pi\alpha' p_R \frac{x^-}{\ell} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n}{n} e^{-\frac{2\pi i n x^-}{\ell}}, \\
\text{where} \quad & [\alpha_m, \alpha_{-n}] = [\tilde{\alpha}_m, \tilde{\alpha}_{-n}] = m\delta_{m,n}, \quad (n, m > 0) \\
& [x_L, p_L] = [x_R, p_R] = i,
\end{aligned}
\tag{2.43}
$$

and all the other commutators vanish. With the periodic boundary condition

$$
\phi(t, x+\ell) = \phi(t,x) + 2\pi R w, \quad w \in \mathbb{Z}, \tag{2.44}
$$

the allowed momentum values are

$$
\begin{aligned}
& p_L = +\frac{R}{\alpha'}w + \frac{k}{R}, \quad p_R = -\frac{R}{\alpha'}w + \frac{k}{R}, \\
& p = \frac{1}{2}(p_L + p_R) = \frac{k}{R}, \quad \tilde{p} = \frac{1}{2}(p_L - p_R) = \frac{R}{\alpha'}w,
\end{aligned}
\tag{2.45}
$$

where k and w are integers. Correspondingly, the chiral boson fields and the dual boson field obey

$$\begin{aligned}
\varphi_L(x+\ell)-\varphi_L(x) &= +\pi\alpha' p_L,\\
\varphi_R(x+\ell)-\varphi_R(x) &= -\pi\alpha' p_R,\\
\phi(x+\ell)-\phi(x) &= \pi\alpha'(p_L-p_R)=2\pi Rw,\\
\theta(x+\ell)-\theta(x) &= \pi\alpha'(p_L+p_R)=2\pi\frac{\alpha'}{R}k.
\end{aligned}\tag{2.46}$$

The set of bosonic exponents consistent with the boundary conditions are $\exp i\left[\frac{k}{R}\phi+\frac{wR}{\alpha'}\theta\right]=\exp i\left[p_L\varphi_L+p_R\varphi_R\right]$. The Hamiltonian is given by

$$\begin{aligned}
H &= H_L+H_R=\frac{2\pi v}{\ell}\left(L_L+L_R\right),\\
L_L &= \frac{\alpha' p_L^2}{4}+\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n-\frac{1}{24},\\
\bar{L}_R &= \frac{\alpha' p_R^2}{4}+\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha_n}-\frac{1}{24}.
\end{aligned}\tag{2.47}$$

Observe that the spectrum depends only on $R/\sqrt{\alpha'}$ and is invariant under $R\to\alpha'/R$ as expected.

2.3.2 Twisted boundary conditions and twisted partition function

Two conserved U(1) charges, one for each left- and right-moving sector, can be introduced as follows:

$$N_{L,R}=\int_0^\ell dx\,\partial_x\varphi_{L,R}=\alpha'\pi p_{L,R},\tag{2.48}$$

which satisfy $[\varphi_L,N_L]=[\varphi_R,N_R]=\alpha'\pi i$. The operator

$$\mathcal{G}(a_c,a_s)=\exp i\left[2\pi a_c R p+2\pi a_s(\alpha'/R)\tilde{p}\right].\tag{2.49}$$

generates translations in ϕ and θ as

$$\phi\to\phi+2\pi a_c R,\quad \theta\to\theta+2\pi a_s(\alpha'/R).\tag{2.50}$$

By using the U(1) × U(1) symmetry generators, it is possible to twist the spatial boundary condition as

$$\begin{aligned}\phi(x+\ell) &= \phi(x) + 2\pi R a_c + 2\pi R w,\\ \theta(x+\ell) &= \theta(x) + 2\pi \frac{\alpha'}{R} a_s + 2\pi \frac{\alpha'}{R} k.\end{aligned} \tag{2.51}$$

With the twisted boundary condition, the momenta are given by

$$\alpha' \tilde{p} = R(a_c + w), \quad \alpha' p = \frac{\alpha'}{R}\left(a_s + k\right). \tag{2.52}$$

As compared to the original quantization conditions, in the presence of the twist, the quantization conditions on p and $\tilde{p}$ are "shifted" by a_c and a_s. Below, we will focus on the twist by the diagonal U(1) symmetry, $a_s = 0$.

Let us now consider the partition function twisted both in time and space directions. It can be written as

$$Z_{[a_c,b_c]} = \mathrm{Tr}_{a_c}\left[\mathcal{G}(b_c) e^{2\pi i\tau(L_0 - c/24) - 2\pi i\bar{\tau}(\bar{L}_0 - c/24)}\right] \tag{2.53}$$

where the trace is taken with the quantization conditions,

$$\begin{aligned}\tilde{p} &= \Delta\tilde{p} + \frac{R}{\alpha'} w, \quad \Delta\tilde{p} = \frac{R a_c}{\alpha'},\\ p &= \Delta p + \frac{k}{R}, \quad \Delta p = \frac{(\alpha'/R) a_s}{\alpha'}.\end{aligned} \tag{2.54}$$

On the other hand, the twist in time direction is implemented as an insertion of the operator $\mathcal{G}(b_c) = \mathcal{G}(b_c, b_s = 0)$.

2.3.3 The CP projected partition function

Following the discussion for fermionic CP symmetric topological insulators, we now project with the CP operator. Inserting the projection operator in the partition function (2.53), we consider

$$Z^{\mathrm{Proj}}_{[a_c]} = \mathrm{Tr}_{a_c}\left[\frac{1+\mathcal{CP}}{2}\mathcal{G}(b_c) q^{(L_0 - c/24)} \bar{q}^{(\bar{L}_0 - c/24)}\right]. \tag{2.55}$$

We will consider an adiabatic process where we change a_c to $a_c + 1$ and ask if the CP projected partition function is invariant or not.

When evaluating the CP projected partition function, it is necessary to know the action of CP operators on the states in the Hilbert space. The symmetry transformation on ϕ and θ implies the action of CP on each mode in the mode expansion of ϕ and θ:

$$\begin{aligned}
&(\mathcal{CP})\alpha_n(\mathcal{CP})^{-1} = \tilde{\alpha}_n, \\
&(\mathcal{CP})\phi_0(\mathcal{CP})^{-1} = -\phi_0, \quad (\mathcal{CP})p(\mathcal{CP})^{-1} = -p, \\
&(\mathcal{CP})\theta_0(\mathcal{CP})^{-1} = \theta_0 + 2\pi\epsilon(\alpha'/R), \quad (\mathcal{CP})\tilde{p}(\mathcal{CP})^{-1} = \tilde{p},
\end{aligned} \tag{2.56}$$

where $\phi_0 = x_L + x_R$ and $\theta_0 = x_L - x_R$. [Since CP flips the sign of p, for generic value of a_s, there is no state that is invariant under a_s. For the purpose of the CP projection, we thus should set $\Delta p = 0 \Rightarrow a_s = 0$.) See discussion near Eq. (2.23).]

For later use, we need to know the action of CP on the states in the zero mode sector. We will use the momentum basis $\{|p,\tilde{p}\rangle\}$, where $|p,\tilde{p}\rangle$ is the momentum eigen ket. Recall that due to the compactification condition, $\phi_0 \equiv \phi_0 + 2\pi R$ and $\theta_0 \equiv \theta_0 + 2\pi\alpha'/R$, the corresponding momenta lie in the BZ. Since CP transformation sends the momentum operators as $p \to -p$ and $\tilde{p} \to \tilde{p}$, the momentum eigen ket $\mathcal{CP}|p,\tilde{p}\rangle$ must be equal to $|-p,\tilde{p}\rangle$ up to a phase, $\mathcal{CP}|p,\tilde{p}\rangle = e^{iA(p,\tilde{p})}|-p,\tilde{p}\rangle$. Similarly, the ket $\mathcal{CP}|\phi_0,\theta_0\rangle$ must be equivalent to $|-\phi_0,\theta_0+2\pi\epsilon(\alpha'/R)\rangle$, $\mathcal{CP}|\phi_0,\theta_0\rangle = e^{iB(\phi_0,\theta_0)}|-\phi_0,\theta_0+2\pi\epsilon(\alpha'/R)\rangle$. We can read off these phases from the Fourier representation of the basis ket:

$$\begin{aligned}
|p,\tilde{p}\rangle &= \int d\phi_0 d\theta_0\, e^{ip\phi_0 + i\tilde{p}\theta_0}|\phi_0,\theta_0\rangle, \\
\text{as} \quad \mathcal{CP}|p,\tilde{p}\rangle &= e^{-i2\pi\epsilon(w+a_c)} \times \int d\phi_0 d\theta_0\, e^{-ip\phi_0 + i\tilde{p}\theta_0} e^{iB}|\phi_0,\theta_0\rangle.
\end{aligned} \tag{2.57}$$

In order to have $\mathcal{CP}|p,\tilde{p}\rangle \propto |-p,\tilde{p}\rangle$, we need to take $B(\phi_0,\theta_0) = \text{const.} = B$, and we conclude

$$\mathcal{CP}|p,\tilde{p}\rangle = e^{-i2\pi\epsilon(w+a_c)}e^{iB}|-p,\tilde{p}\rangle, \tag{2.58}$$

i.e., $A(p,\bar{p}) = B - 2\pi\epsilon\tilde{p}\alpha'/R$. One can also check, from the inverse Fourier representation

$$|\phi_0,\theta_0\rangle = \sum_{p,\tilde{p}} e^{-ip\phi_0 - i\tilde{p}\theta_0}|p,\tilde{p}\rangle, \tag{2.59}$$

that $\mathcal{CP}|\phi_0,\theta_0\rangle = e^{iB}|-\phi_0,\theta_0+2\pi\epsilon\alpha'/R\rangle$.

Summarizing, acting with CP operator on the basis ket $|p,\tilde{p}\rangle$,

$$\mathcal{CP}|p,\tilde{p}\rangle = P_{[a_c]}e^{-i2\pi\epsilon w}|-p,\tilde{p}\rangle, \tag{2.60}$$

where $P_{[a_c]}$ is independent of p and $\tilde{p}$, but may be dependent on the adiabatic parameters, a_c and b_c. While the presence of the phase factor $e^{-i2\pi\epsilon w}$ can directly be seen from the CP transformation laws on the zero mode operators, the phase factor $P_{[a_c]}$ cannot be determined; this originates from our ignorance on the CP eigenvalue of the ground state, and, in particular, on its dependence on the adiabatic parameters. Based on our previous discussion, however, we enforce CP invariance for arbitrary value of the adiabatic parameters. This means that we demand that our CP operation does not depend on the inserted flux. While the Hilbert space changes adiabatically, we do not allow the phase factor to be dependent on a_c. This is the same assumption we had before for the case of fermions. We could then choose $B=\pi a_c$, and hence $P_{[a_c]}=1$.

Having established the CP action on the zero-mode wave functions, we now calculate the projected partition function explicitly. Note also $(\mathcal{CP})\mathcal{G}(b_c)(\mathcal{CP})^{-1}=\mathcal{G}(-b_c)$. This limits a reasonable value of $2\pi Rb_c$ to be 0 and π. Then, the Klein bottle partition function is

$$\begin{aligned} Z^{\text{Klein}}_{[a_c]} &= \mathrm{Tr}_{a_c}\left[(\mathcal{CP})\mathcal{G}(b_c)q^{(L_0-c/24)}\bar{q}^{(\bar{L}_0-c/24)}\right] \\ &= (q\bar{q})^{-\frac{1}{24}}\prod_{n=1}[1-(q\bar{q})^n]^{-1}\sum_{p,\tilde{p}}\langle p\tilde{p}|(\mathcal{CP})e^{iRb_c\frac{1}{2}(p_L+p_R)}q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2}|p\tilde{p}\rangle. \end{aligned} \tag{2.61}$$

In order for $\langle p\tilde{p}|\mathcal{CP}|p\tilde{p}\rangle$ to be non-zero, p must be zero ($k=0$). Then, $p_L=-p_R=\tilde{p}$ and hence,

$$Z^{\text{Klein}}_{[a_c]} = (q\bar{q})^{-\frac{1}{24}}\prod_{n=1}[1-(q\bar{q})^n]^{-1}\sum_{w\in\mathbb{Z}}(q\bar{q})^{\frac{1}{4}\left(\frac{R}{\sqrt{\alpha'}}\right)^2(w+a_c)^2}e^{-i2\pi\epsilon w}. \tag{2.62}$$

When $\epsilon=1/2$, the partition function is not invariant under $a_c\to a_c+1$, as it picks up an overall minus sign, $Z^{\text{Klein}}_{[a_c+1]}=-Z^{\text{Klein}}_{[a_c]}$. As in the fermionic CP symmetric topological insulator, the anomaly is of $\mathbb{Z}_2$ kind since it vanishes when we consider two copies (or any even number of copies) of the theory.

2.4 K-matrix theories protected by symmetries

In this section, based upon the previous sections, we consider edge theories consisting of multiple free bosons that can describe, in addition to non-interacting topological insulators, interacting Abelian topologically ordered phases. We will develop a criterion for the stability of the edge theories in the presence of CP and U(1) symmetries.

2.4.1 K-matrix theories

Let us consider the K-matrix theory with N component compactified boson fields described by the Lagrangian

$$\mathcal{L} = \frac{1}{4\pi}\left(K_{IJ}\partial_t\phi^I\partial_x\phi^J - V_{IJ}\partial_x\phi^I\partial_x\phi^J\right) + \frac{e}{2\pi}\epsilon^{\mu\nu}Q_I\partial_\mu\phi^I A^c_\nu + \frac{s}{2\pi}\epsilon^{\mu\nu}S_I\partial_\mu\phi^I A^s_\nu, \tag{2.63}$$

where K is an $N\times N$ symmetric and invertible matrix with integer-valued matrix elements, V is an $N\times N$ symmetric and positive definite matrix that accounts for the (screened) translation-invariant two-body interactions between electrons. The N component vector ("charge vector") Q_I, together with the unit of electric charge e, describe how the system couples to an external electromagnetic U(1) gauge potential, A^c_μ. Similarly, the N component vector ("spin vector") S_I, together with the unit of "spin" charge s, describe how the system couples to an external "spin" U(1) gauge potential A^s_μ that couples to the spin-1/2 degrees of freedom along some quantization axis, z-axis, say.

The boson fields are compact variables, meaning field configurations ϕ^I differ by an integer multiple of 2π are identified:

$$\phi^I(t,x) \equiv \phi^I(t,x) + 2\pi n^I, \tag{2.64}$$

with $n^I \in \mathbb{Z}$ for all $I = 1,\dots,N$. The equal-time canonical commutation relations of the boson fields are given by [8]

$$[\phi^I(t,x), \partial_x\phi^J(t,x')] = -2\pi i(K^{-1})^{IJ}\delta(x-x'), \tag{2.65}$$

or equivalently

$$[\phi^I(t,x), \phi^J(t,x')] = -i\pi\left[(K^{-1})^{IJ}\mathrm{sgn}(x-x') + \Theta^{IJ}\right], \tag{2.66}$$

where the Klein factor

$$\Theta^{IJ} := (K^{-1})^{IK}\left[\mathrm{sgn}(K-L)(K_{KL}+Q_KQ_L)\right](K^{-1})^{LJ} \tag{2.67}$$

is included to ensure that local excitations satisfy the proper commutation relations.

[8] Here and in the following, the Dirac delta function $\delta(x-x')$ and $\mathrm{sgn}\,(x-x')$ in the commutator should be interpreted as its periodic counter part, such as $\sum_{m\in\mathbb{Z}}\delta(x-x'-2m\pi)$, when the system is put on a circle of circumference 2π.

The goal of this section is to develop, in the presence of either charge or spin U(1) symmetry, together with a discrete symmetry (such as CP or parity symmetry), a stability ("ingappability") criterion of the edge theory (2.63) against interactions.

The rotated basis We start our discussion by quantizing the K-matrix theory with the (untwisted) compactificataion condition (2.64). We introduce, starting from the original boson fields $\{\phi^I\}_{I=1,\dots,N}$, a new basis $\{\varphi^i\}_{i=1,\dots,N}$ that is obtained by a rotation matrix e^i_I and its inverse $e^{\star J}_j$ as

$$\begin{aligned} &\varphi^i \equiv e^i_I \phi^I, \quad \phi^J \equiv e^{\star J}_j \varphi^j, \\ &e^i_I e^{\star J}_i = \delta^J_I, \quad e^i_I e^{\star I}_j = \delta^i_j. \end{aligned} \tag{2.68}$$

The "vielbein" e^i_I and $e^{\star J}_j$ diagonalize the K-matrix as

$$K_{IJ} = e^i_I \eta_{ij} e^j_J, \quad \eta_{ij} = e^{\star I}_i K_{IJ} e^{\star J}_j = \eta_i \delta_{ij} \tag{2.69}$$

where $\eta_{ij} = \eta_i \delta_{ij}$ is a diagonal matrix. We also note

$$(K^{-1})^{IJ} = e^{\star I}_i (\eta^{-1})^{ij} e^{\star J}_j. \tag{2.70}$$

In the following, by choosing e^i_I and $e^{\star I}_i$ properly, we assume that η_i's are ± 1. In the rotated basis φ, the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{4\pi}\left(\eta_{ij}\partial_t \varphi^i \partial_x \varphi^j - v\delta_{ij}\partial_x \varphi^i \partial_x \varphi^j\right) + \frac{e}{2\pi}\epsilon^{\mu\nu}\tilde{Q}_i \partial_\mu \varphi^i A_\nu + \frac{s}{2\pi}\epsilon^{\mu\nu}\tilde{S}_i \partial_\mu \varphi^i B_\nu, \tag{2.71}$$

where we have introduced the charge and spin vectors in the rotated basis as

$$\tilde{Q}_i \equiv e^{\star I}_i Q_I, \quad \tilde{S}_i \equiv e^{\star I}_i S_I, \tag{2.72}$$

and assumed, for simplicity, $e^{\star I}_i V_{IJ} e^{\star J}_j = v_{ij} = v_i \delta_{ij}$. The compactification condition in the original basis (2.64) is translated into, in the rotated basis,

$$\varphi^i(t,x) \equiv \varphi^i(t,x) + 2\pi e^i_I n^I. \tag{2.73}$$

Quantization without the background fields As a warm up, we first quantize canonically the theory without the background fields on the spatial circle of radius 2π:

$$\mathcal{L}_0 = \frac{1}{4\pi}\left(\eta_{ij}\partial_t\varphi^i\partial_x\varphi^j - v_{ij}\partial_x\varphi^i\partial_x\varphi^j\right). \tag{2.74}$$

The equal-time commutation relations are

$$[\varphi^i(t,x), \partial_x\varphi^j(t,x')] = -2\pi i(\eta^{-1})^{ij}\delta(x-x'). \tag{2.75}$$

The mode expansion of φ is given by

$$\varphi^i(t,x) = \varphi_0^i - p_j\left[(\eta^{-1})^{jk}v_{kl}(\eta^{-1})^{li}t + (\eta^{-1})^{ji}x\right] + i\sum_{n\neq 0} b_{nj}e^{-in\left[(\eta^{-1})^{jk}v_{kl}(\eta^{-1})^{li}t+(\eta^{-1})^{ji}x\right]}, \tag{2.76}$$

together with the commutation relation

$$[\varphi_0^i, p_j] = i\delta_j^i, \quad [b_{ni}, b_{mj}] = \frac{1}{m}\delta_{ij}\delta_{n+m}. \tag{2.77}$$

(All other commutators vanish.)

As p_i is conjugate to $\varphi^i(t,x)$, which obeys the compactification condition $\varphi^i(t,x) \simeq \varphi^i(t,x) + 2\pi e_I^i n^I$, the quantization condition of p_i is given in terms of the reciprocal lattice vectors $e_i^{\star I}$ as

$$p_i = e_i^{\star I} m_I, \quad m_I \in \mathbb{Z}^N. \tag{2.78}$$

I.e., while the coordinates φ^i are compactified on a lattice Γ spanned by $\{e^i\}$, the momenta p_i lie in the reciprocal (dual) lattice $\tilde{\Gamma}$ spanned by $\{e_i^\star\}$. Observe also that, in a momentum eigenstate, the mode expansion (2.76) implies the boundary condition

$$\varphi^i(t,x+2\pi) = \varphi^i(t,x) - 2\pi p_j(\eta^{-1})^{ij} = \varphi^i(t,x) + 2\pi e_j^{\star J} m_J(\eta^{-1})^{ij} = \varphi^i(t,x) + 2\pi e_I^i(K^{-1})^{IJ}m_J. \tag{2.79}$$

For generic integral values of m_J, $(K^{-1})^{IJ}m_J$ are not integers and hence the boson fields obey twisted boundary conditions. The states corresponding to the momentum p_j are represented by (by state-operator correspondence) the vertex operators

$$\ddagger \exp ip_i\varphi^i(t,x)\ddagger = \ddagger \exp im_I\phi_I(t,x)\ddagger \tag{2.80}$$

where $\ddagger \cdots \ddagger$ represents normal-ordering.

Let us consider a subset of $\tilde{\Gamma}$, that is obtained by choosing $m_I = K_{IJ}\Lambda^J$ with $\Lambda^J \in \mathbb{Z}^N$. For this choice, the momentum is given by

$$p_i = \eta_{ij} e^j_J \Lambda^J \tag{2.81}$$

and the boson fields φ^i obey untwisted boundary conditions. In the sector of the theory with this choice of momentum, all excitations are local (excitations consisting of exciting electron-like particles). The corresponding vertex operators are

$$\ddagger \exp i\Theta(\boldsymbol{\Lambda})\ddagger = \ddagger \exp i\Lambda^I K_{IJ}\phi_J(t,x) \ddagger . \tag{2.82}$$

To summarize, quantization of the K-matrix theory with the compactification conditions (2.63-2.64) gives rise to the spectrum of local (electrons) as well as non-local (quasiparticle) excitations, which are represented by untwisted and twisted boundary conditions, respectively. Once we specify the boundary condition by some integer vector $\boldsymbol{m}$, we obtain the spectrum quantized within one sector (labeled by the equivalent class $[\boldsymbol{m}]$ with the relation $\boldsymbol{m} \equiv \boldsymbol{m} + K\boldsymbol{\Lambda}$) of the total spectrum. There are $|\det K|$ sectors in this compactified K-matrix theory.

The Hamiltonian and total momentum are

$$\begin{aligned} H_0 &= \frac{1}{4\pi}\int_0^{2\pi} dx \partial_x \varphi^i v_{ij} \partial_x \varphi^j \\ &= \frac{1}{2}\left(\eta^{-1}p\right)^i v_{ij}\left(\eta^{-1}p\right)^j - \frac{1}{24}\mathrm{tr}\left(\eta^{-1}v\eta^{-1}\right) + \sum_{n=1}^{\infty} n^2 \left(\eta^{-1}b_{-n}\right)^i v_{ij}\left(\eta^{-1}b_n\right)^j \end{aligned} \tag{2.83}$$

and

$$\begin{aligned} P_0 &= \frac{1}{4\pi}\int_0^{2\pi} dx \partial_x \varphi^i \eta_{ij} \partial_x \varphi^j \\ &= \frac{1}{2} p_i (\eta^{-1})^{ij} p_j - \frac{1}{24}\mathrm{tr}\left(\eta^{-1}\right) + \sum_{n=1}^{\infty} n^2 b_{-ni}(\eta^{-1})^{ij} b_{nj}, \end{aligned} \tag{2.84}$$

respectively. The eigenstates of H_0 and P_0 can be expressed as a direct product of their oscillator part (the Fock states generated by b_{ni}) and non-oscillator part (related to φ^i_0 and p_i). For the non-oscillator part, one can choose to use the momentum eigenvalues $\{p_i\}$, which have values $\{\eta_{ij}e^j_J\Lambda^J + e^{\star I}_i m_I\}$ as the boundary condition $\phi^I(t,x+2\pi) = \phi^I(t,x) + 2\pi(K^{-1})^{IJ}m_J$ [or $\varphi^i(t,x+2\pi) = \varphi^i(t,x) + 2\pi e^i_I (K^{-1})^{IJ} m_J$ in the rotated basis] is specified, to label the eigenstates. We denote these eigenstates of sector $[\boldsymbol{m}]$ as

$|\boldsymbol{\Lambda}_{\boldsymbol{m}}\rangle \equiv |\boldsymbol{\Lambda} + K^{-1}\boldsymbol{m}\rangle$.

The partition function for the sector $[\boldsymbol{m}]$ evaluated on a torus with modular parameter $\tau = \tau_1 + i\tau_2$ is given by

$$Z_{\boldsymbol{m}}(\tau) = \mathrm{Tr}_{\boldsymbol{m}}\left[e^{-2\pi i\tau_1 P_0} e^{-2\pi\tau_2 H_0}\right]. \tag{2.85}$$

Twisted boundary conditions by U(1) symmetries

The K-matrix theory (2.63) has $\mathrm{U}(1)^N$ symmetries. The corresponding conserved charges are given by

$$\mathcal{C}^I \equiv \frac{1}{2\pi}\int_0^{2\pi} dx\partial_x\phi^I = -2\pi(K^{-1})^{IJ}e^j_J p_j. \tag{2.86}$$

The global U(1) transformations associated to these charge degrees of freedom are generated by

$$\mathcal{G}(\boldsymbol{\alpha}) \equiv e^{-2\pi i\alpha_I\mathcal{C}^I} \quad \text{as} \quad \mathcal{G}(\boldsymbol{\alpha})\varphi^i(t,x)\mathcal{G}(\boldsymbol{\alpha})^{-1} = \varphi^i(t,x) + 2\pi(\eta^{-1})^{ij}e^{\star J}_j\alpha_J, \tag{2.87}$$

where $\boldsymbol{\alpha}$ is a vector consisting of twisting phases.

Now, starting from the original boundary condition for sector $[\boldsymbol{m}]$ (2.79), we can generate a new twisted boundary condition by acting with $\mathcal{G}$ as

$$\phi^I(t,x+2\pi) = \mathcal{G}(\boldsymbol{a})\phi^I(t,x)\mathcal{G}(\boldsymbol{a})^{-1} + 2\pi(K^{-1})^{IJ}m_J = \phi^I(t,x) + 2\pi(K^{-1})^{IJ}(m_J + a_J), \tag{2.88}$$

or, in the rotated basis,

$$\varphi^i(t,x+2\pi) = \mathcal{G}(\boldsymbol{a})\varphi^i(t,x)\mathcal{G}(\boldsymbol{a})^{-1} + 2\pi(\eta^{-1})^{ij}e^{\star J}_j m_J = \varphi(t,x) + 2\pi(\eta^{-1})^{ij}e^{\star J}_j\left(m_J + a_J\right). \tag{2.89}$$

With this twisted boundary condition, the allowed values of the momenta p are now shifted and given by

$$p_i = e^{\star I}_i\left(m_I + K_{IJ}\Lambda^J + a_I\right) \equiv e^{\star I}_i K_{IJ}\Lambda^J_{\boldsymbol{m}+\boldsymbol{a}}, \tag{2.90}$$

where

$$\Lambda^J_{\boldsymbol{m}+\boldsymbol{a}} := \Lambda^J + (K^{-1})^{JI}\left(m_I + a_I\right),\ \Lambda^J \in \mathbb{Z}^N. \tag{2.91}$$

As in the untwisted case [in the absence of U(1) twisting phases], the eigenstates of the Hamiltonian and the total momentum can be expressed as a direct product of their oscillator part and non-oscillator part. One can choose to use the momentum eigenvalues, which are specified by a set of integers $\Lambda^J \in \mathbb{Z}^N$ to label non-oscillator part of the eigenstates. We denote these basis states as $|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$, which are given by the untwisted eigenstates with $\boldsymbol{m}$ shifted by $\boldsymbol{a}$.

Twisted partition function

The twisted partition function for sector $[\boldsymbol{m}]$ evaluated on a torus with modular parameter $\tau = \tau_1 + i\tau_2$ is given by

$$Z_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}(\tau) = \mathrm{Tr}_{\boldsymbol{m}+\boldsymbol{a}} \left[\mathcal{G}(\boldsymbol{b}) e^{-2\pi i \tau_1 P_0} e^{-2\pi\tau_2 H_0}\right], \tag{2.92}$$

where the trace is taken over the Hilbert space in the presence of the twisted boundary condition generated by $\mathcal{G}(\boldsymbol{a})$. The operator insertion $\mathcal{G}(\boldsymbol{b})$ generates, in the path-integral picture, twisted boundary condition in time direction. The partition function can be expressed as a product of the oscillator part and the zero-mode part as

$$Z_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}(\tau) = \xi(\tau) \sum_{\boldsymbol{\Lambda}\in\mathbb{Z}^N} \zeta^{\boldsymbol{\Lambda}}_{[\boldsymbol{m}+\boldsymbol{a},\boldsymbol{b}]}(\tau) \langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle, \tag{2.93}$$

where

$$\zeta^{\boldsymbol{\Lambda}}_{[\boldsymbol{m}+\boldsymbol{a},\boldsymbol{b}]}(\tau) \equiv \exp\Big(2\pi i \boldsymbol{b}^T \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} - \pi i \tau_1 \boldsymbol{\Lambda}^T_{\boldsymbol{m}+\boldsymbol{a}} K \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} - \pi\tau_2 \boldsymbol{\Lambda}^T_{\boldsymbol{m}+\boldsymbol{a}} V \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\Big). \tag{2.94}$$

The oscillator part of the partition function $\xi(\tau)$ is independent of the twisting angles $\boldsymbol{a}$ and $\boldsymbol{b}$ and will not play any important role in the following discussion. The overlap $\langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ in Eq. (2.93) is simply $\langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle = 1$, but we displayed $\langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ in Eq. (2.93) for the later comparison.

Large gauge transformations

The large gauge transformations of U(1) symmetries are finite gauge transformations that preserve the spectrum of the theory. They are finite shifts of twisting phases $\boldsymbol{a}$ and $\boldsymbol{b}$ that preserve the U(1) operators $\mathcal{G}$ [or more precisely, preserve the (twisted) boundary conditions] and can be deduced from the compactification condition of the K-matrix theory (2.64). For $\mathrm{U}(1)^N$ symmetry, the large gauge transformations are given by

$$\boldsymbol{a} \to \boldsymbol{a} + K\boldsymbol{\delta}, \quad \boldsymbol{b} \to \boldsymbol{b} + K\boldsymbol{\delta}', \quad \forall \boldsymbol{\delta}, \boldsymbol{\delta}' \in \mathbb{Z}^N. \tag{2.95}$$

To discuss the behavior of the twisted partition function under the large gauge transformation, let us consider $\widetilde{\mathrm{U}(1)}^2 = \mathrm{U}(1)_c \times \mathrm{U}(1)_s$ symmetry: $\boldsymbol{a} = a_c\boldsymbol{Q} + a_s\boldsymbol{S}$ and $\boldsymbol{b} = b_c\boldsymbol{Q} + b_s\boldsymbol{S}$, where $\boldsymbol{Q}$ and $\boldsymbol{S}$ are charge and spin vectors, respectively. The minimal shifts are given by

$$\delta_c = 1/\beta_c, \quad \delta_s = 1/\beta_s, \tag{2.96}$$

where $\beta_c \equiv \min_{\boldsymbol{l}} |\boldsymbol{l}^T K^{-1}\boldsymbol{Q}|$ and $\beta_s \equiv \min_{\boldsymbol{l}} |\boldsymbol{l}^T K^{-1}\boldsymbol{S}|$ represent the elementary charge and spin (the smallest fractional charge and spin of quasiparticle excitations) of the system, respectively. In other words, *classically*, the system is expected to be invariant under the following large gauge transformation:

$$a_{c/s} \to a_{c/s} + \delta_{c/s}, \quad b_{c/s} \to b_{c/s} + \delta_{c/s}. \tag{2.97}$$

The invariance under the large gauge transformation may, however, be violated at the quantum level. From Eq. (2.93), we see, under large gauge transformations for the charge $\mathrm{U}(1)_c$ symmetry,

$$\begin{aligned} Z_{\boldsymbol{m}[a_c+\delta_c, b_c, a_s, b_s]} &= Z_{\boldsymbol{m}[a_c, b_c, a_s, b_s]}, \\ Z_{\boldsymbol{m}[a_c, b_c+\delta_c, a_s, b_s]} &= e^{2\pi i \delta_c \boldsymbol{Q}^T K^{-1}(a_c\boldsymbol{Q}+a_s\boldsymbol{S})} \cdot Z_{\boldsymbol{m}[a_c, b_c, a_s, b_s]}. \end{aligned} \tag{2.98}$$

Similarly, under large gauge transformations for the spin $\mathrm{U}(1)_s$ symmetry,

$$\begin{aligned} Z_{\boldsymbol{m}[a_c, b_c, a_s+\delta_s, b_s]} &= Z_{\boldsymbol{m}[a_c, b_c, a_s, b_s]}, \\ Z_{\boldsymbol{m}[a_c, b_c, a_s, b_s+\delta_s]} &= e^{2\pi i \delta_s \boldsymbol{S}^T K^{-1}(a_c\boldsymbol{Q}+a_s\boldsymbol{S})} \cdot Z_{\boldsymbol{m}[a_c, b_c, a_s, b_s]}. \end{aligned} \tag{2.99}$$

Observe that the way the partition function changes under the large gauge transformations does not depend on the sector $\boldsymbol{m}$ we specify.

In the cases where there is only the charge U(1) symmetry, the above calculation tells us that $Z_{[a_c,b_c]}(\tau)$ is not invariant under the large-gauge transformations if

$$\boldsymbol{Q}^T K^{-1}\boldsymbol{Q} \neq 0. \tag{2.100}$$

This large gauge anomaly is nothing but the quantum Hall effect.

2.4.2 Symmetry projected partition functions: generalities

Now let us move on to the situations of our main interest. We consider the K-matrix theory that preserves one of the U(1) symmetries, $\mathrm{U}(1)_c$ or $\mathrm{U}(1)_s$, but not both. We denote this U(1) symmetry by $\mathscr{G} = \mathrm{U}(1)_{c,s}$. In addition, we assume the K-matrix theory is invariant under yet another global unitary symmetry; we call the corresponding symmetry group $\mathscr{G}'$. In our examples below, $\mathscr{G}'$ consists of a single discrete unitary symmetry transformation such as CP or P transformation. The total symmetry group is $\mathscr{G} \times \mathscr{G}'$ or $\mathscr{G} \rtimes \mathscr{G}'$. While our methodology applied to either case equally, we will focus on the case where the total symmetry group is given by the semi direct product $\mathscr{G} \rtimes \mathscr{G}'$, for which case we have found topologically non-trivial (i.e., anomalous) cases.

Under the action of a symmetry generator $\mathcal{M} \in \mathscr{G}'$, the bosonic fields transform as:

$$\mathcal{M}\boldsymbol{\phi}(t,x)\mathcal{M}^{-1} = U_M \boldsymbol{\phi}(t, r_M x) + \pi K^{-1}\boldsymbol{\chi}_M, \tag{2.101}$$

where U_M is an integral $N \times N$ matrix, r_M is a real number, and $\boldsymbol{\chi}_M$ is some N-component real vector. For on-site symmetry, $r_M = 1$. For non-on-site symmetry (below we consider parity, P, or some on-site symmetry combined with parity, such as CP), we have $r_M = -1$. Assuming the K-matrix theory is invariant under group $\mathscr{G}'$, U_M and r_M must satisfy

$$U_M^T K U_M = r_M K, \quad U_M^T V U_M = r_M^2 V = V, \tag{2.102}$$

for any $\mathcal{M} \in \mathscr{G}'$. The invariance under $\mathscr{G}'$ also imposes constraints on the integer vector $\boldsymbol{Q}$ or $\boldsymbol{S}$ through the way the charge or spin current are transformed under $\mathscr{G}'$.

Following our discussion in the previous sections, our strategy to diagnose the stability of the edge theory is to enforce the invariance under $\mathscr{G}'$ by projection, and discuss the dependence of the projected partition function on the twisting phases. In order for this strategy to work, the twisted boundary conditions should be invariant under the symmetry $\mathscr{G}'$. Acting on the twisted boundary condition with a symmetry generator (2.88),

$$\begin{aligned}
&\mathcal{M}K\boldsymbol{\phi}(t, x+2\pi)\mathcal{M}^{-1} = \mathcal{M}K\boldsymbol{\phi}(t,x)\mathcal{M}^{-1} + 2\pi\left(\boldsymbol{m} + \boldsymbol{a}\right) \\
\Rightarrow\ &K\boldsymbol{\phi}(t, r_M(x+2\pi)) = K\boldsymbol{\phi}(t, r_M x) + 2\pi r_M U_M^T \left(\boldsymbol{m} + \boldsymbol{a}\right).
\end{aligned} \tag{2.103}$$

In order for the twisted boundary condition (2.88) to be invariant under $\mathcal{M}$ for arbitrary value of a_c (a_s), the charge vector $\boldsymbol{Q}$ (the spin vector $\boldsymbol{S}$) must satisfy

$$U_M^T \boldsymbol{m} = \boldsymbol{m} \quad \text{and} \quad U_M^T \boldsymbol{Q} = \boldsymbol{Q} \quad (U_M^T \boldsymbol{S} = \boldsymbol{S}), \tag{2.104}$$

respectively. In our discussion below, we assume, for given $\mathscr{G}$ and $\mathscr{G}'$, this condition is satisfied.

Finally, from the group structure of $\mathscr{G}'$ and the statistics of vertex operators, which represent local excitations, there are further constrains on the possible form of U_M and $\boldsymbol{\chi}_M$. This issue will be discussed in more details later with specific examples. In conclusion, a general K-matrix theory with symmetry group $\mathscr{G} \rtimes \mathscr{G}'$ is described by the data $\{K, \boldsymbol{Q} \,\text{or}\, \boldsymbol{S}, \{U_M, \boldsymbol{\chi}_M | \mathcal{M} \in \mathscr{G}'\}\}$ that satisfies the conditions discussed above.

The symmetry projected partition function for the sector $[\boldsymbol{m}]$ is defined by

$$Z^{\text{Proj}}_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]} \equiv \text{Tr}_{\boldsymbol{m}+\boldsymbol{a}} \left[\mathcal{P}_{\mathscr{G}'} \mathcal{G}(\boldsymbol{b}) e^{-2\pi i \tau_1 P_0} e^{-2\pi \tau_2 H_0} \right], \quad \text{where} \quad \mathcal{P}_{\mathscr{G}'} = |\mathscr{G}'|^{-1} \sum_{\mathcal{M} \in \mathscr{G}'} \mathcal{M} \tag{2.105}$$

is the projection operator for the symmetry group $\mathscr{G}'$, satisfying $\mathcal{P}^2_{\mathscr{G}'} = \mathcal{P}_{\mathscr{G}'}$. The trace in Eq. (2.105) is taken with respect to the Hilbert space in the presence of boundary conditions twisted by $\mathcal{G}(\boldsymbol{a})$, and the insertion of the operator $\mathcal{G}(\boldsymbol{b})$ inside the trace represents, in the path integral picture, the U(1) twisting phase in the temporal direction. As mentioned earlier, the twisting should be invariant under $\mathscr{G}'$, and hence typically only the charge twisting angles $[a_c, b_c]$ or the spin twisting angles $[a_s, b_s]$ is allowed. In this section, we discuss some general properties of $Z^{\text{Proj}}_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}$ keeping both charge and spin twisting angles. Once $\mathscr{G}'$ is given, and the invariance of the twisting boundary condition by $\mathscr{G}'$ is taken into account, it is easy to "switch off" either one of charge or spin angle.

The twisted partition function, that appears as a part of the projected partition function $Z^{\text{Proj}}_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}$, can be evaluated as

$$Z^{M}_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}(\tau) = \text{Tr}_{\boldsymbol{m}+\boldsymbol{a}} \left[\mathcal{M} \mathcal{G}(\boldsymbol{b}) e^{-2\pi i \tau_1 P_0} e^{-2\pi \tau_2 H_0} \right] = \xi^M(\tau) \sum_{\boldsymbol{\Lambda} \in \mathbb{Z}^N} \zeta^{\boldsymbol{\Lambda}}_{[\boldsymbol{m}+\boldsymbol{a},\boldsymbol{b}]} \langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} | \mathcal{M} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} \rangle, \tag{2.106}$$

where the oscillator part of the partition function $\xi^M(\tau)$ does not depend on the twisting phases $\boldsymbol{a}$ and $\boldsymbol{b}$. The most crucial part of the calculations, as inferred from the previous examples of the Dirac fermions and the single-component boson, is the matrix element $\langle \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{m}+\boldsymbol{a}} | \mathcal{M} | \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} \rangle$ in Eq. (2.106). As one can read off from Eq. (2.101), the transformation $\mathcal{M}$ maps the momentum eigenvalues $\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} \to r_M U_M \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}$, and hence $\mathcal{M}|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ should be equal to $|r_M U_M \boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ up to a phase factor.

To calculate this phase factor, in particular in the presence of the Klein factors, it is convenient to use the state-operator correspondence; according to the state-operator correspondence, for each sector of the

Hilbert space constructed out of the zero-mode $|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$, we have a corresponding operator

$$\ddagger \exp i\boldsymbol{\Lambda}^T_{\boldsymbol{m}+\boldsymbol{a}} K\boldsymbol{\phi} \ddagger . \tag{2.107}$$

As a warm up, let us consider the untwisted ($\boldsymbol{a}=0$) counterpart when $\boldsymbol{m}=0$

$$\ddagger \exp i\boldsymbol{\Lambda}^T K\boldsymbol{\phi} \ddagger . \tag{2.108}$$

Now, using symmetry conditions (2.102) we have

$$\begin{aligned}\mathcal{M} \ddagger e^{i\boldsymbol{\Lambda}^T K\boldsymbol{\phi}(t,0)} \ddagger \mathcal{M}^{-1} &= e^{i\Delta\phi^{\boldsymbol{\Lambda}}_M} \ddagger e^{i\boldsymbol{\Lambda}^T K\left(\mathcal{M}\boldsymbol{\phi}(t,0)\ \mathcal{M}^{-1}\right)} \ddagger \\ &= e^{i\Delta\phi^{\boldsymbol{\Lambda}}_M} \ddagger e^{i\boldsymbol{\Lambda}^T K\left(U_M\boldsymbol{\phi}(t,0)+\pi K^{-1}\boldsymbol{\chi}_M\right)} \ddagger \\ &= e^{i\Delta\phi^{\boldsymbol{\Lambda}}_M} e^{i\pi\boldsymbol{\Lambda}^T\boldsymbol{\chi}_M} \ddagger e^{i(r_M U_M\boldsymbol{\Lambda})^T K\boldsymbol{\phi}(t,0)} \ddagger,\end{aligned} \tag{2.109}$$

where $e^{i\Delta\phi^{\boldsymbol{\Lambda}}_M}$ is the statistical phase factor of the vertex operator $\ddagger e^{i\boldsymbol{\Lambda}^T K\boldsymbol{\phi}}\ddagger$ under symmetry transformation $\mathcal{M}$, as explained in Appendix A.2. For bosonic systems, such phase factor $e^{i\Delta\phi^{\boldsymbol{\Lambda}}_M}$ equals to 1 because of the commutativity among bosons. For fermionic systems, however, we must take into account the anti-commutativity among fermions, which may lead to an additional phase factor for the transformation of the vertex operator.

In the presence of the twisting angles, the action of $\mathcal{M}$ on non-oscillator state $|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ may give rise to an additional phase factor which in principle depends on $\boldsymbol{m}+\boldsymbol{a}$. Let us now take a close look at this. States $|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ labeled by shifted momentum $\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}$ can be viewed as generated from a ground state $|\mathrm{GS}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ by acting on some raising operator. By assumption, $|\mathrm{GS}_{\boldsymbol{m}+\boldsymbol{a}}\rangle$ is invariant under $\mathcal{M}$, and hence

$$\mathcal{M}|\mathrm{GS}_{\boldsymbol{m}+\boldsymbol{a}}\rangle = P^M_{[\boldsymbol{m}+\boldsymbol{a}]}|\mathrm{GS}_{\boldsymbol{m}+\boldsymbol{a}}\rangle, \tag{2.110}$$

where the eigenvalue of $\mathcal{M}$ for the ground state, denoted by $P^M_{[\boldsymbol{m}+\boldsymbol{a}]}$, depends on the spatial twisting phases. The phase $P^M_{[\boldsymbol{m}+\boldsymbol{a}]}$ together with $e^{i\left[\Delta\phi^{\boldsymbol{\Lambda}}_M+\pi\boldsymbol{\Lambda}^T\boldsymbol{\chi}_M\right]}$ in Eq. (2.109), leads to

$$\mathcal{M}|\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle = P^M_{[\boldsymbol{m}+\boldsymbol{a}]} e^{i\left[\Delta\phi^{\boldsymbol{\Lambda}}_M+\pi\boldsymbol{\Lambda}^T\boldsymbol{\chi}_M\right]}|r_M U_M\boldsymbol{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}\rangle. \tag{2.111}$$

In other words, the dependence on the twisting angles comes only from $P^M_{[\boldsymbol{m}+\boldsymbol{a}]}$, but not from $e^{i\left[\Delta\phi^{\boldsymbol{\Lambda}}_M+\pi\boldsymbol{\Lambda}^T\boldsymbol{\chi}_M\right]}$. In fact, as we noted previously in the case of the Dirac fermion and the single-component boson, once we insist on invariance under $\mathscr{G}'$, the eigenvalues of the symmetry transformation would not change as we change

$a_{s,c}$. If so, the phase $e^{i\left[\Delta\phi_M^{\mathbf{\Lambda}}+\pi\mathbf{\Lambda}^T\boldsymbol{\chi}_M\right]}$, which we get from the vertex operator of the untwisted theory, is the only phase factor that we need to keep track of.

Since $\mathcal{M}$ maps the momentum eigenvalues $\mathbf{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} \to r_M U_M \mathbf{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}$, in the summation in Eq. (2.106), only those $\mathbf{\Lambda}$s that satisfy $\mathbf{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}} = r_M U_M \mathbf{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}$ contribute. With the conditions (2.102) and (2.104), this means that the first term in $\mathbf{\Lambda}_{\boldsymbol{m}+\boldsymbol{a}}$ in Eq. (2.91) satisfies $\mathbf{\Lambda} = r_M U_M \mathbf{\Lambda}$. Then the twisted partition function (2.106) is given by

$$Z^M_{\boldsymbol{m}[\boldsymbol{a},\boldsymbol{b}]}(\tau) = \xi^M(\tau) P^M_{[\boldsymbol{m}+\boldsymbol{a}]} \sum_{\substack{\mathbf{\Lambda}\in\mathbb{Z}^N,\\ \mathbf{\Lambda}=r_M U_M\mathbf{\Lambda}}} e^{i\left[\Delta\phi_M^{\mathbf{\Lambda}}+\pi\mathbf{\Lambda}^T\boldsymbol{\chi}_M\right]} \zeta^{\mathbf{\Lambda}}_{[\boldsymbol{m}+\boldsymbol{a},\boldsymbol{b}]}(\tau). \tag{2.112}$$

From Eq. (2.112) we observe that the symmetry projected partition function for the sector $[\boldsymbol{m}]$ depends only on parameters $\boldsymbol{m}+\boldsymbol{a}$ and $\boldsymbol{b}$. This means that the way the projected partition function changes under large gauge transformation does not depend on $\boldsymbol{m}$ (i.e., it is independent of sector). For compactness, we will drop the label $\boldsymbol{m}$ (or just set $\boldsymbol{m}=\mathbf{0}$) on partition functions in the following discussion.

2.4.3 $\mathscr{G} \rtimes \mathscr{G}' = \mathrm{U}(1)_c \rtimes Z_2^{\mathrm{CP}}$

Now we consider the non-on-site CP symmetry. Let

$$(\mathcal{CP})\boldsymbol{\phi}(t,x)(\mathcal{CP})^{-1} = U_{\mathrm{CP}}\boldsymbol{\phi}(t,-x) + \pi K^{-1}\boldsymbol{\chi}_{\mathrm{CP}}, \tag{2.113}$$

where U_{CP} is an integer $N\times N$ matrix (the same as the dimension of K) and $\boldsymbol{\chi}_{\mathrm{CP}}$ is some N-component real vector. In order for the system to be CP invariant, we require

$$\begin{aligned} &U_{\mathrm{CP}}^T K U_{\mathrm{CP}} = -K, \quad U_{\mathrm{CP}}^T V U_{\mathrm{CP}} = V, \quad U_{\mathrm{CP}}^2 = I_N, \\ &U_{\mathrm{CP}}^T \boldsymbol{Q} = \boldsymbol{Q}, \quad (I_N - U_{\mathrm{CP}}^T)\boldsymbol{\chi}_{\mathrm{CP}} = 2\epsilon\boldsymbol{Q} \mod 2, \end{aligned} \tag{2.114}$$

where I_N is the $N\times N$ identity matrix and the value $\epsilon = 0, 1/2$ represents the sign of the CP operator squared for fermionic systems, with the relation

$$(\mathcal{CP})^2 = e^{i2\pi\epsilon N_f}, \tag{2.115}$$

where N_f is the total fermion number operator.

In fact, these constraints on $(K, \boldsymbol{Q}, U_{\mathrm{CP}}, \boldsymbol{\chi}_{\mathrm{CP}})$ are identical to the corresponding data in K-matrix theories with time-reversal invariance [90]. The most general gauge inequivalent solution (which exists for a non-chiral

K-matrix theory; N must be even) is of the form

$$K = \begin{pmatrix} 0 & A & B & B \\ A^T & 0 & C & -C \\ B^T & C^T & \Gamma & W \\ B^T & -C^T & W^T & -\Gamma \end{pmatrix}, \ \boldsymbol{Q} = \begin{pmatrix} 0 \\ \boldsymbol{q'} \\ \boldsymbol{q} \\ \boldsymbol{q} \end{pmatrix},$$

$$U_{\mathrm{CP}} = \begin{pmatrix} -I_M & 0 & 0 & 0 \\ 0 & I_M & 0 & 0 \\ 0 & 0 & 0 & I_{N/2-M} \\ 0 & 0 & I_{N/2-M} & 0 \end{pmatrix}, \quad \boldsymbol{\chi}_{\mathrm{CP}} = \begin{pmatrix} \boldsymbol{x} \\ 0 \\ (1-2\epsilon)\boldsymbol{x'} \\ (1-2\epsilon)\boldsymbol{x'} + 2\epsilon\boldsymbol{q} \end{pmatrix}. \tag{2.116}$$

Here, the matrix A is $M \times M$, while the matrices B, C are $M \times (N-M)$. The matrices Γ, W are both $(N/2-M) \times (N/2-M)$. Similarly, $\boldsymbol{q'}$ is of dimension M and $\boldsymbol{Q}$ is of dimension $(N/2-M)$. Finally, $\boldsymbol{x}$ is a M-dimensional vector consisting of 1's and 0's, while $\boldsymbol{x'}$ is a $(N/2-M)$-dimensional vector consisting of 1's and 0's. There are only a few constraints on $(A, B, C, \Gamma, W, \boldsymbol{q}, \boldsymbol{q'}, \boldsymbol{x}, \boldsymbol{x'})$. First, W must be antisymmetric: $W = -W^T$. This requirement follows from CP symmetry (2.114). Second, $\boldsymbol{q'}$ must be even-valued. This constraint comes from $Q_I = K_{II} \mod 2$, which means the insulator is composed out of electrons. For the same reason, the parity of Q_I must match with that of K_{II}, but can be either even or odd. Finally, the greatest common factor of $\{Q_I\}$ must be 1.

Once these data are given, we now calculate the CP symmetry projected partition function with charge U(1) symmetry ($\boldsymbol{a} = a_c\boldsymbol{Q}$, $\boldsymbol{b} = b_c\boldsymbol{Q}$):

$$Z^{\mathrm{Proj}}_{[a_c,b_c]}(\tau) = \mathrm{Tr}_{a_c}\left[\mathcal{P}_{\mathrm{CP}}\mathcal{G}(b_c)e^{-2\pi i\tau_1 P_0}e^{-2\pi\tau_2 H_0}\right], \quad \text{with} \quad \mathcal{P}_{\mathrm{CP}} = \frac{1+\mathcal{CP}}{2}. \tag{2.117}$$

Bosonic systems

In the case where the system is composed of bosons, the most general data $(K, \boldsymbol{Q}, U_{\mathrm{CP}}, \boldsymbol{\chi}_{\mathrm{CP}})$ is given by (2.116), with an additional condition that the charge vector $\boldsymbol{Q}$ is even valued (and thus the diagonal elements of Γ are also even). In this CP invariant theory, the function $\zeta^{\boldsymbol{\Lambda}}_{[a_c,b_c]}(\tau)$ with the constraint $\boldsymbol{\Lambda} = -U^T_{\mathrm{CP}}\boldsymbol{\Lambda}$ is given by

$$\zeta^{\boldsymbol{\Lambda}}_{[a_c,b_c]}(\tau) = \exp\left(-\pi\tau_2\boldsymbol{\Lambda}^T_{a_c}V\boldsymbol{\Lambda}_{a_c}\right), \tag{2.118}$$

where $\mathbf{\Lambda}_{a_c} \equiv \mathbf{\Lambda} + a_c K^{-1}\boldsymbol{Q}$ and the fact $\boldsymbol{Q}^T \mathbf{\Lambda}_{a_c} = \mathbf{\Lambda}_{a_c}^T K \mathbf{\Lambda}_{a_c} = 0$ (by CP symmetry) is used. Therefore, the partition function

$$Z^{\mathrm{CP}}_{[a_c,b_c]} = \mathrm{Tr}_{a_c}\left[(\mathcal{CP})\mathcal{G}(b_c)e^{-2\pi i\tau_1 P_0}e^{-2\pi\tau_2 H_0}\right] \tag{2.119}$$

is calculated as [Eq. (2.112)]

$$Z^{\mathrm{CP}}_{[a_c,b_c]}(\tau) = P^{\mathrm{CP}}_{[a_c]}\xi^{\mathrm{CP}}(\tau) \sum_{\substack{\mathbf{\Lambda}\in\mathbb{Z}^N \\ \mathbf{\Lambda}=-U^T_{\mathrm{CP}}\mathbf{\Lambda}}} e^{-\pi\tau_2 \mathbf{\Lambda}^T_{a_c}\mathbf{\Lambda}_{a_c} + i\pi\mathbf{\Lambda}^T\chi_{\mathrm{CP}}}, \tag{2.120}$$

where $P^{\mathrm{CP}}_{[a_c]}$ is the CP eigenvalue of the ground state. Observe that the charge U(1) transformation operator $\mathcal{G}(b_c) = e^{-2\pi i b_c N_f}$ and the spatial translation operator (in space coordinate x) $e^{-2\pi i\tau_1 P_0}$ in the partition function are both projected out, leading to the independence of a_c and τ_1 in Z^{CP}. This can also be argued by the fact that the total charge J^0_c and momentum P_0 are odd under CP, while the Hamiltonian H_0 is even. For the same reason, the function ξ^{CP} just depends on τ_2. [See similar discussion near Eqs. (2.36) and (2.61).]

The bosonic CP symmetry projected partition function is given by

$$Z^{\mathrm{Proj}}_{[a_c,b_c]}(\tau) = \frac{1}{2}\left[Z_{[a_c,b_c]}(\tau) + Z^{\mathrm{CP}}_{[a_c]}(\tau_2)\right], \tag{2.121}$$

with the form of $Z_{[a_c,b_c]}(\tau)$ given by (2.93). Under a large gauge transformation $a_c \to a_c + \delta_c$ and $b_c \to b_c + \delta_c$, where $\delta_c \equiv (\min_{\boldsymbol{l}} |\boldsymbol{l}^T K^{-1}\boldsymbol{Q}|)^{-1}$, we have

$$\begin{aligned} Z_{[a_c+\delta_c,b_c]}(\tau) &= Z_{[a_c,b_c]}(\tau), \\ Z^{\mathrm{CP}}_{[a_c+\delta_c]}(\tau_2) &= \frac{P^{\mathrm{CP}}_{[a_c+\delta_c]}}{P^{\mathrm{CP}}_{[a_c]}} \cdot e^{-i\pi\mathbf{\Lambda}_c^T\boldsymbol{\chi}_{\mathrm{CP}}} \cdot Z^{\mathrm{CP}}_{[a_c]}(\tau_2), \end{aligned} \tag{2.122}$$

where $\mathbf{\Lambda}_c \equiv \delta_c K^{-1}\boldsymbol{Q}$ is an integer vector, and

$$Z_{[a_c,b_c+\delta_c]}(\tau) = e^{2\pi i\delta_c a_c \boldsymbol{Q}^T K^{-1}\boldsymbol{Q}} \cdot Z_{[a_c,b_c]}(\tau). \tag{2.123}$$

Since $\boldsymbol{Q}^T K^{-1}\boldsymbol{Q} = 0$ by CP symmetry, the total projected partition function is invariant under $b_c \to b_c + \delta_c$. The crucial part is the behavior of the partition function under $a_c \to a_c + \delta_c$. If we demand the CP eigenvalue be invariant under $a_c \to a_c + \delta_c$, *i.e.*, $P^{\mathrm{CP}}_{[a_c+\delta_c]} = P^{\mathrm{CP}}_{[a_c]}$, then the partition function is (not) large gauge invariant if the value of $\mathbf{\Lambda}_c^T \boldsymbol{\chi}_{\rm CP}$ is even (odd). Therefore, the quantity

$$\mathbf{\Lambda}_c^T \boldsymbol{\chi}_{\rm CP} = \delta_c \boldsymbol{\chi}_{\rm CP}^T K^{-1} \boldsymbol{Q} \tag{2.124}$$

gives the criterion: "$\mathbf{\Lambda}_c^T \boldsymbol{\chi}_{\rm CP} =$ odd number" corresponds to theory with anomaly (topological phase), while "$\mathbf{\Lambda}_c^T \boldsymbol{\chi}_{\rm CP} =$ even number" corresponds to theory without anomaly (trivial phase).

Fermionic systems

For fermionic systems, the most general data $(K, \boldsymbol{Q}, U_{\rm CP}, \boldsymbol{\chi}_{\rm CP})$ is given by

$$\begin{aligned} K &= \begin{pmatrix} \Gamma & W \\ W^T & -\Gamma \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{q} \end{pmatrix}, \\ U_{\rm CP} &= \begin{pmatrix} 0 & I_{N/2} \\ I_{N/2} & 0 \end{pmatrix}, \quad \boldsymbol{\chi}_{\rm CP} = \begin{pmatrix} 2(1/2-\epsilon)\boldsymbol{x}' \\ 2(1/2-\epsilon)\boldsymbol{x}' + 2\epsilon\boldsymbol{q} \end{pmatrix}. \end{aligned} \tag{2.125}$$

The calculation of the CP symmetry projected partition function in this theory can be done in the same way as the bosonic case, except the statistical phase factors that arise in:

$$(\mathcal{CP}) \ddagger e^{i\Theta(\mathbf{\Lambda})} \ddagger (\mathcal{CP})^{-1} = e^{i\Delta\phi_{\rm CP}^{\mathbf{\Lambda}}} e^{i\pi \mathbf{\Lambda}^T \boldsymbol{\chi}_{\rm CP}} \ddagger e^{i\Theta(-U_{\rm CP}\mathbf{\Lambda})} \ddagger, \tag{2.126}$$

where

$$i\Delta\phi_{\rm CP}^{\mathbf{\Lambda}} = i\pi \left(\sum_{I=1}^{N/2} \Lambda_I Q_I \right) \left(\sum_{J=N/2+1}^{N} \Lambda_J Q_J \right) \mod 2\pi i \tag{2.127}$$

is the statistical phase factor due to Fermi statistics derived in Appendix A.2. For CP invariant vectors $\mathbf{\Lambda}$ satisfying $\mathbf{\Lambda} = -U_{\rm CP}\mathbf{\Lambda}$, we can express $\mathbf{\Lambda}$ as $(\boldsymbol{\lambda}, -\boldsymbol{\lambda})^T$, where $\boldsymbol{\lambda}$ is an $N/2$ dimensional integer vector. Then the statistical phase can be expressed as

$$i\Delta\phi_{\rm CP}^{\mathbf{\Lambda}} = -i\pi \sum_{I=1}^{N/2} \lambda_I q_I = -i\pi \boldsymbol{\lambda}^T \boldsymbol{q} \mod 2\pi i. \tag{2.128}$$

On the other hand,

$$i\pi \mathbf{\Lambda}^T \boldsymbol{\chi}_{\rm CP} = -i\epsilon\pi \boldsymbol{\lambda}^T \boldsymbol{q} \mod 2\pi i. \tag{2.129}$$

Writing $\boldsymbol{\Lambda}_{a_c}^T \boldsymbol{\Lambda}_{a_c} = 2\boldsymbol{\lambda}_{a_c}^T \boldsymbol{\lambda}_{a_c}$, where $\boldsymbol{\lambda}_{a_c} \equiv \boldsymbol{\lambda} + \frac{a_c}{\delta_c}\boldsymbol{\lambda}_c$ and $\boldsymbol{\lambda}_c$ is defined as

$$\delta_c K^{-1}\boldsymbol{Q} = \boldsymbol{\Lambda}_c \equiv \begin{pmatrix} \boldsymbol{\lambda}_c \\ -\boldsymbol{\lambda}_c \end{pmatrix}, \tag{2.130}$$

(remember that $\boldsymbol{\Lambda}_c = -U_{\mathrm{CP}}\boldsymbol{\Lambda}_c$, so $\boldsymbol{\lambda}_c$ is well-defined), then we have

$$\begin{aligned} Z^{\mathrm{CP}}_{[a_c,b_c]}(\tau = P^{\mathrm{CP}}_{[a_c]}\xi^{\mathrm{CP}}(\tau) &\sum_{\substack{\boldsymbol{\Lambda}\in\mathbb{Z}^N \\ \boldsymbol{\Lambda}=-U^T_{\mathrm{CP}}\boldsymbol{\Lambda}}} e^{-\pi\tau_2\boldsymbol{\Lambda}^T_{a_c}\boldsymbol{\Lambda}_{a_c}+i\pi\boldsymbol{\Lambda}^T\chi_{\mathrm{CP}}+i\Delta\phi^{\boldsymbol{\Lambda}}_{CP}} \\ = P^{\mathrm{CP}}_{[a_c]}\xi^{\mathrm{CP}}(\tau) &\sum_{\boldsymbol{\lambda}\in\mathbb{Z}^{N/2}} e^{-2\pi\tau_2\boldsymbol{\lambda}^T_{a_c}\boldsymbol{\lambda}_{a_c}-2\pi i(\epsilon+1/2)\boldsymbol{\lambda}^T\boldsymbol{q}}. \end{aligned} \tag{2.131}$$

As in the case of the bosonic systems discussed previously, here $Z^{\mathrm{CP}}_{[a_c,b_c]}(\tau)$ depends only on a_c and τ_2. The fermionic CP symmetry projected partition functions are given by

$$Z^{\mathrm{Proj}}_{[a_c,b_c]}(\tau) = \frac{1}{2}\left[Z_{[a_c,b_c]}(\tau) + Z^{\mathrm{CP}}_{[a_c]}(\tau_2)\right] \tag{2.132}$$

with $Z_{[a_c,b_c]}(\tau)$ given by Eq. (2.93). Under the large gauge transformation $a_c \to a_c + \delta_c$ and $b_c \to b_c + \delta_c$

$$\begin{aligned} Z_{[a_c+\delta_c,b_c]}(\tau) &= Z_{[a_c,b_c+\delta_c]}(\tau) = Z_{[a_c,b_c]}(\tau), \\ Z^{\mathrm{CP}}_{[a_c+\delta_c]}(\tau_2) &= \frac{P^{\mathrm{CP}}_{[a_c+\delta_c]}}{P^{\mathrm{CP}}_{[a_c]}} \cdot e^{i2\pi(\epsilon-1/2)\boldsymbol{\lambda}^T_c\boldsymbol{q}} \cdot Z^{\mathrm{CP}}_{[a_c]}(\tau_2), \end{aligned} \tag{2.133}$$

where $\boldsymbol{Q}^T K^{-1}\boldsymbol{Q} = 0$ (by CP symmetry) is used. Therefore, the fermionic theory is always anomaly-free if $\epsilon = 1/2$ [$(\mathcal{CP})^2 = (-1)^{N_f}$]. For $\epsilon = 0$ [$(\mathcal{CP})^2 = 1$], the quantity $\boldsymbol{\lambda}_c^T\boldsymbol{q}$ gives the stability criterion: "$\boldsymbol{\lambda}_c^T\boldsymbol{q} =$ odd number" corresponds to an anomalous theory (topological phase), while "$\boldsymbol{\lambda}_c^T\boldsymbol{q} =$ even number" corresponds to theory without anomaly (trivial phase):

$$\begin{aligned} \boldsymbol{\lambda}_c^T\boldsymbol{q} = \text{odd} \quad &\Longrightarrow \quad \text{stable edge ("topological")}, \\ \boldsymbol{\lambda}_c^T\boldsymbol{q} = \text{even} \quad &\Longrightarrow \quad \text{unstable edge ("trivial")}. \end{aligned} \tag{2.134}$$

2.4.4 Examples

The double Laughlin edge state

As an example, let us now consider the case of the doubled fermionic Laughlin state described by

$$K = \begin{pmatrix} \frac{1}{\nu} & 0 \\ 0 & -\frac{1}{\nu} \end{pmatrix}, \quad U_{\mathrm{CP}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\chi}_{\mathrm{CP}} = \begin{pmatrix} 0 \\ 2\epsilon \end{pmatrix}, \tag{2.135}$$

where ν^{-1} is an odd integer and ϵ can be either 0 or 1/2. The elementary charge in the system is $\beta_c = \min_{\boldsymbol{l}} |\boldsymbol{l}^T K^{-1} \boldsymbol{Q}| = \nu$, and the quantity $\boldsymbol{\Lambda}_c$ is given by $\boldsymbol{\Lambda}_c = K^{-1}\boldsymbol{Q}/m_c = (1,-1)^T = (\boldsymbol{\lambda}_c, -\boldsymbol{\lambda}_c)^T$. From the previous discussion, the criterion for topological phases is: if $\epsilon = 1/2$, the system is in the trivial phase; if $\epsilon = 0$, since $\boldsymbol{\lambda}_c^T \boldsymbol{q} = 1$, the system is in the topological phase.

In this theory, we have $\xi^{\mathrm{CP}}(\tau) = \eta(2i\nu\tau_2)^{-1}$ and thus Z^{CP} is given by [Eq. (2.131)]

$$\begin{aligned} Z^{\mathrm{CP}}_{[a_c]}(\tau) &= \frac{P^{\mathrm{CP}}_{[a_c]}}{\eta(2i\nu\tau_2)} \sum_{\boldsymbol{\lambda}\in\mathbb{Z}} e^{-2\pi\tau_2(\boldsymbol{\lambda}+\nu a_c)^2 - i2\pi(\epsilon+1/2)\boldsymbol{\lambda}} \\ &= e^{2\pi i\nu a_c(\epsilon-1/2)} \frac{P^{\mathrm{CP}}_{[a_c]}}{\eta(2i\nu\tau_2)} \vartheta \begin{bmatrix} \nu a_c \\ -(\epsilon-1/2) \end{bmatrix} (0, 2i\tau_2). \end{aligned} \tag{2.136}$$

The total CP symmetry projected partition function is given by Eq. (2.132). For $\nu = 1$, which corresponds to the "integer" CP symmetric system (without ground-state degeneracy), the results here agree exactly with the CP projected partition function obtained for the free fermion theory. Under a large gauge transformation $a_c \to a_c + 1/\nu$, we have

$$Z^{\mathrm{CP}}_{[a_c+\frac{1}{\nu}]}(\tau_2) = e^{2\pi i(\epsilon-1/2)} \cdot \frac{P^{\mathrm{CP}}_{[a_c+\frac{1}{\nu}]}}{P^{\mathrm{CP}}_{[a_c]}} \cdot Z^{\mathrm{CP}}_{[a_c]}(\tau_2). \tag{2.137}$$

Alternatively, the stability of the edge state of this theory can also be analyzed by enumerating potential (interaction) terms that can potentially gap the edge state without breaking CP and charge U(1) symmetries. They are given by

$$U(x)\cos\left[\Theta(\boldsymbol{\Lambda}) - \alpha(x)\right] = U(x)\cos\left[\frac{n}{\nu}(\phi_1+\phi_2) - \alpha(x)\right], \tag{2.138}$$

where $U(x)$ and $\alpha(x)$ represent the strength and phase of the potential, respectively, which are allowed to be spatially inhomogeneous, and $\boldsymbol{\Lambda}^T = (n, -n)$, $n \in \mathbb{Z}$ is a charge conserving vector. Under the CP transformation,

$$(\mathcal{CP})\left[U(x)\cos\left(\Theta(\boldsymbol{\Lambda})-\alpha(x)\right)\right](\mathcal{CP})^{-1}=U(x)\cos\left[\Theta(\boldsymbol{\Lambda})-(2\epsilon+1)n\pi-\alpha(x)\right], \tag{2.139}$$

where Eqs. (2.126) and (2.127) are used. For $\epsilon=1/2$, the scattering term is CP invariant for any integer n. Such perturbation can gap out the edge without breaking CP symmetry of the ground state $\frac{1}{\nu}\langle\phi_1+\phi_2\rangle$. On the other hand, for $\epsilon=0$ the scattering term is CP invariant just for even n. In this case, however, the gapping perturbation also spontaneously breaks the CP symmetry of the ground state: $\frac{1}{\nu}\langle\phi_1+\phi_2\rangle\to\frac{1}{\nu}\langle\phi_1+\phi_2\rangle-\pi$. The argument here agrees with our generalized Laughlin argument based on the CP projected partition function.

The fermionic 4×4 K-matrix theory

As yet another example, let us consider the fermionic K-matrix theory described by the following 4×4 K-matrix:

$$K=\begin{pmatrix}\Gamma & 0\\ 0 & -\Gamma\end{pmatrix},\quad U_{\mathrm{CP}}=\begin{pmatrix}0 & I_2\\ I_2 & 0\end{pmatrix},\quad \boldsymbol{Q}=(1,1,1,1)^T,\quad \boldsymbol{\chi}_{\mathrm{CP}}=(0,0,2\epsilon,2\epsilon)^T, \tag{2.140}$$

where Γ is a 2×2 matrix. It is convenient to parameterize the matrix as [90]

$$K=\begin{pmatrix}b+us & b & 0 & 0\\ b & b+vs & 0 & 0\\ 0 & 0 & -b-us & -b\\ 0 & 0 & -b & -b-vs\end{pmatrix}, \tag{2.141}$$

where b,u,v,s are integers and u and v have no common factor. In terms of these parameters, the elementary charge is

$$\beta_c=\min_{\boldsymbol{l}}|\boldsymbol{l}^TK^{-1}\boldsymbol{Q}|=\frac{1}{(u+v)b+uvs}, \tag{2.142}$$

and the quantities $\mathbf{\Lambda}_c$ and $\boldsymbol{\lambda}_c$ are given by

$$\mathbf{\Lambda}_c = \frac{1}{m_c} K^{-1} \boldsymbol{Q} = \begin{pmatrix} v \\ u \\ -v \\ -u \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}_c \\ -\boldsymbol{\lambda}_c \end{pmatrix}. \tag{2.143}$$

From these, the criterion for the presence/absence of SPT phases is: if $\epsilon = 1/2$, the system is always in the trivial phase; if $\epsilon = 0$, since $\boldsymbol{\lambda}_c^T \boldsymbol{q} = u + v$, the parity of $u + v$ determines whether the phase is trivial ($u + v$ is even) or topological ($u + v$ is odd).

The CP twisted partition function Z^{CP} is given by [Eq. (2.131)]

$$Z^{\mathrm{CP}}_{[a_c]}(\tau) = \xi^{\mathrm{CP}}(\tau) P^{\mathrm{CP}}_{[a_c]} \sum_{\lambda_1, \lambda_2 \in \mathbb{Z}} e^{-2\pi\tau_2 \left[(\lambda_1 + m_c a_c v)^2 + (\lambda_2 + m_c a_c u)^2\right]} \times e^{-2\pi i (\epsilon + 1/2)(\lambda_1 + \lambda_2)}. \tag{2.144}$$

Under a large gauge transformation $a_c \to a_c + 1/\beta_c$, the CP-twisted partition function transforms as

$$Z^{\mathrm{CP}}_{[a_c + 1/\beta_c]}(\tau_2) = e^{i2\pi(\epsilon + 1/2)(u+v)} \cdot \frac{P^{\mathrm{CP}}_{[a_c + 1/\beta_c]}}{P^{\mathrm{CP}}_{[a_c]}} \cdot Z^{\mathrm{CP}}_{[a_c]}(\tau_2). \tag{2.145}$$

We can also look for gapping potentials and see if we can gap the edge without breaking CP and charge U(1) symmetries [89]. To gap out the 4 edge modes of theory (2.140), we need to find two linearly independent and charge conserving vectors $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ that satisfy Haldane's null vector criterion:

$$\mathbf{\Lambda}_1^T K \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2^T K \mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^T K \mathbf{\Lambda}_2 = 0. \tag{2.146}$$

Such $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ can be the following cases:

$\mathbf{\Lambda}_1 = U_{\mathrm{CP}}^T \mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2 = -U_{\mathrm{CP}}^T \mathbf{\Lambda}_2$: In this case, the charge conserving conditions $\mathbf{\Lambda}_1^T \boldsymbol{Q} = \mathbf{\Lambda}_2^T \boldsymbol{Q} = 0$ and Eq. (2.146) give that

$$\begin{aligned} \mathbf{\Lambda}_1 &= n_1 (1, -1, 1, -1)^T \equiv n_1 \mathbf{\Lambda}_-, \\ \mathbf{\Lambda}_2 &= n_2 (v, u, -v, -u)^T \equiv n_2 \mathbf{\Lambda}_+, \end{aligned} \tag{2.147}$$

where $n_1, n_2 \in \mathbb{Z}$. Under CP the scattering term $\sum_{i=1}^{2} U_i(x)\cos[\Theta(\mathbf{\Lambda}_i) - \alpha_i(x)]$ transforms as

$$
\begin{aligned}
&(\mathcal{CP})\left[\sum\nolimits_{i=1}^{2} U_i(x)\cos\left[\Theta(\mathbf{\Lambda}_i) - \alpha_i(x)\right]\right](\mathcal{CP})^{-1} \\
&= U_1(x)\cos\left[-\Theta(\mathbf{\Lambda}_1) - \alpha_1(x)\right] + U_2(x)\cos\left[\Theta(\mathbf{\Lambda}_2) - n_2(2\epsilon+1)(u+v)\pi - \alpha_2(x)\right].
\end{aligned} \tag{2.148}
$$

By choosing $\alpha_1(x) = k\pi,\ k = 0, 1$, then, for $\epsilon = 1/2$, the scattering term is CP invariant for any integers n_1 and n_2. Such perturbation can gap out the edge without breaking CP symmetry of the ground state $\{\langle\Theta(\mathbf{\Lambda}_-)\rangle, \langle\Theta(\mathbf{\Lambda}_+)\rangle\}$. For $\epsilon = 0$, the scattering term is CP invariant if $n_2(u+v)$ is even. Under CP transformation the ground state transforms as

$$
\{\langle\Theta(\mathbf{\Lambda}_-)\rangle, \langle\Theta(\mathbf{\Lambda}_+)\rangle\} \to \{-\langle\Theta(\mathbf{\Lambda}_-)\rangle, \langle\Theta(\mathbf{\Lambda}_+)\rangle - (u+v)\pi\}. \tag{2.149}
$$

Therefore, the perturbation will gap out the edge with ($u+v$ is odd)/without ($u+v$ is even) breaking the CP symmetry of the ground state spontaneously.

$\mathbf{\Lambda}_2 = -U_{\mathrm{CP}}^T\mathbf{\Lambda}_1$: In this case the CP invariant scattering term is

$$
U(x)\big[\cos\left(\Theta(\mathbf{\Lambda}_1) - \alpha(x)\right) + \cos(\Theta(\mathbf{\Lambda}_2) + \pi\mathbf{\Lambda}_1^T\chi + \Delta\phi_{\mathrm{CP}}^{\mathbf{\Lambda}_1} - \alpha(x))\big], \tag{2.150}
$$

where we used the fact that $\mathbf{\Lambda}_i^T\boldsymbol{Q} = 0$ (the charge neutrality condition) and $\Delta\phi_{\mathrm{CP}}^{\mathbf{\Lambda}_1} = \Delta\phi_{\mathrm{CP}}^{\mathbf{\Lambda}_2}$. Defining $\mathbf{\Lambda}'_\pm = \mathbf{\Lambda}_1 \pm \mathbf{\Lambda}_2$, we can then find that $\mathbf{\Lambda}'_+$ is an integer multiple of $(v, u, -v, -u)$ and $\mathbf{\Lambda}'_-$ is an integer multiple of $(1, -1, 1, -1)$. From the analysis in (i), we know that $\langle\Theta(v, u, -v, -u)\rangle$ spontaneously breaks CP for odd $u+v$ ($\epsilon = 0$), so it is impossible that $\langle\Theta(\mathbf{\Lambda}_1)\rangle$ and $\langle\Theta(\mathbf{\Lambda}_2)\rangle$ (thus $\langle\Theta(\mathbf{\Lambda}_1) + \Theta(\mathbf{\Lambda}_2)\rangle$) can condensate without spontaneously breaking CP symmetry. The result for $\epsilon = 1/2$ is the same in (i): the perturbation can gap out the edge without breaking CP symmetry of the ground state. On the other hand, for even $u+v$ we can take

$$
\mathbf{\Lambda}'^T_- = (1, -1, 1, -1), \quad \mathbf{\Lambda}'^T_+ = (v, u, -v, -u), \tag{2.151}
$$

so that

$$
\begin{aligned}
\mathbf{\Lambda}_1^T &= \frac{1}{2}(1+v, -1+u, 1-v, -1-u), \\
\mathbf{\Lambda}_2^T &= \frac{1}{2}(-1+v, 1+u, -1-v, 1-u).
\end{aligned} \tag{2.152}
$$

Under CP the ground state transforms as

$$\{\langle\Theta(\mathbf{\Lambda}_1)\rangle\,,\langle\Theta(\mathbf{\Lambda}_2)\rangle\}\rightarrow\{\langle\Theta(\mathbf{\Lambda}_2)\rangle-(u+v)(\epsilon+1/2)\pi,\langle\Theta(\mathbf{\Lambda}_1)\rangle-(u+v)(\epsilon+1/2)\pi\},\tag{2.153}$$

which does not break CP spontaneously. Also, since there are no non-primitive linear combinations [9] $a_1\mathbf{\Lambda}_1+a_2\mathbf{\Lambda}_2$ for any integers a_1 and a_2, the perturbation does not break CP for the whole family of condensations $\langle\Theta(a_1\mathbf{\Lambda}_1+a_2\mathbf{\Lambda}_2)\rangle$ (for both $\epsilon=0,1/2$). Therefore, the argument of the stability of the edge state by the microscopic analysis is consistent with the one by the large gauge invariance of the CP symmetry-projected partition function.

2.5 Discussion

We have developed a general theoretical framework that allows us to determine under which conditions a given edge theory is gappable/ingappable. While we have worked out particular examples with CP or P symmetry, our theoretical framework is applicable to other examples with local and non-local symmetries. For example, our methodology can be applicable to reflection symmetric fermionic SPT phases that are classified and tabulated in Refs. [75, 76].

We conclude with the following three comments:

(i) The role of conformal symmetry. For the examples we have looked at, the edge states are given by a CFT which enjoys holomorphic - anti-holomorphic factorization. The fact that the Virasoro and current algebras act as a spectrum generating algebra makes calculations of the partition functions of the edge theories (with CP projection) tractable. However, we anticipate the proposed scheme to diagnose parity symmetric (non-) SPT phases has a wider applicability than edge theories described by a CFT, as we argue below. First of all, while we are not to be restricted to relativistic systems in condensed matter physics, some universal physical properties of general, non-relativistic systems at long wavelength limit, such as the band topology or the electromagnetic response, are often encoded in topological field theories. Since topological, these theories respect the Lorentz symmetry and in fact are conformal. In addition, from the perspective of topological classification of states of matter, classifying SPT phases of non-interacting fermion systems, for example, can be done solely in terms of Dirac operators (which is Lorentz invariant) with symmetry restrictions.

More fundamentally, just like the original Laughlin's argument does not refer the holomorphic-anti-

[9] The linear combination $a_1\mathbf{\Lambda}_1+a_2\mathbf{\Lambda}_2$ is non-primitive if there are some integer vector $\mathbf{\Lambda}$ and integer $k>1$ such that $a_1\mathbf{\Lambda}_1+a_2\mathbf{\Lambda}_2=k\mathbf{\Lambda}$.

holomorphic factorization (but keeping in mind that, in practice, universal properties of edge theories of many quantum Hall systems are described by a CFT), our methodology may well be applicable to edge theories without conformal symmetry. In fact, our method itself does not mention any underlying conformal or Lorentz symmetries. Only addition to Laughlin's argument is the symmetry-projection.

(ii) Our consideration in this paper is limited to an anomaly associated to global U(1) symmetries (once CP is enforced) and thus limited to systems with conserved U(1) symmetry (such as conservation of the particle number or z-component of SU(2) spin). We use a gauge flux of these U(1) symmetries as an adiabatic parameter in developing Laughlin's gauge argument. For systems that do not have such continuous symmetry, one may need to consider an anomaly associated to gravitational degrees of freedom such as modular invariance [28]. As one can see from the fact that the real part of the modular parameter τ_1 is projected out by orientifolding [recall discussion near Eq. (2.36)], the modular group of the torus $\mathrm{PSL}(2,\mathbb{Z})$ cannot be used to study conformal field theories on the Klein bottle. Nevertheless, an analogue of the S-modular transformation (which exchanges the space and time direction on the torus) still plays a role in orientifold conformal field theories. To be more precise, the "loop channel" calculations presented in this paper can be cast into an equivalent calculation in the "tree channel" by using crosscap states.We plan to visit these tree channel pictures in a forthcoming publication.

(iii) Related to this question, in this paper, we discussed partition functions of edge theories on the Klein bottle, but not the other spacetime manifolds such as annulus or Möbius strip. It may be interesting to ask if there is any role played by these other geometries. In oriented cases, the modular invariance on the torus is believed to be enough to define the conformal field theory on any (oriented) world sheet. For the unoriented cases, one may wonder if considering the consistency of the theory on the Klein bottle would be enough to define the conformal field theory on all (unoriented) worldsheets.

It is worth pointing out the following: in string theory, conformal field theories on the Klein bottle appear in Type I superstring theory. There are, in addition to the Klein bottle, other worldsheet geometries such as a strip and the Möbius strip. Some properties are physically constrained by tadpole cancellation, and worldsheet geometries with boundaries come hand-in-hand with D-branes. It would be interesting to explore what role these and other consistency conditions might play in a condensed matter setting.

Chapter 3

Fermionic symmetry-protected topological phases in 3+1 dimensions: Global anomalies of surface theories on unorientable three-manifolds

3.1 Introduction

Basing on the previous results of (2+1)d SPT phases, we study the surface states of (3+1)d fermionic SPT phases from the perspective of global quantum anomalies in this chapter. We will discuss two examples: (i) the Dirac fermion surface state of (3+1)d bulk CP symmetric topological insulators (TIs) and (ii) the Majorana fermion surface state of (3+1)d bulk reflection symmetric crystalline topological superconductors (TSCs). The bulk phase of the first example is a fermionic SPT phase protected by electromagnetic U(1) and CP [product of charge conjugation and mirror reflection (parity)] symmetries. This example is CPT-conjugate to a (3+1)d time-reversal symmetric TI (class AII), and characterized by a $\mathbb{Z}_2$ topological number [111]. The bulk phase of the second example is a fermionic SPT phase protected by fermion number parity and reflection (parity) symmetry. It belongs to symmetry class D+R_+ crystalline TSCs [1] in Refs. [75, 76, 112]. This example is CPT-conjugate to a (3+1)d time-reversal symmetric TSC (class DIII) [111]. At non-interacting level, class D+R_+ crystalline TSCs in (3+1)d are characterized by an integer ($\mathbb{Z}$) topological

This chapter was written based on the result of a previous publication [55] of the dissertation author and collaborators.

[1] The subscript "+" in symmetry class D+R_+ indicates that the two symmetry operations, charge-conjugation (or particle-hole) and reflection symmetries, of the single-particle Hamiltonians in this symmetry class commute with each other.

number (the Mirror Chern number), similar to their CPT partner, class DIII TSCs [2]. On the other hand, a number of recent works showed that the integral non-interacting classification of class DIII TSCs breaks down to $\mathbb{Z}_{16}$ once interactions are included [48, 49, 41, 50, 51, 52, 53]. Such collapses have been also reported in one and two spatial dimensions [24, 25, 26, 46, 27, 28, 29, 47].

Similar to the approach in chapter ??, we enforce CP or reflection symmetry on the surface theories by taking an orientifold projection [78, 79, 113, 80, 81, 82]. In the first example, the resulting projected theory is then shown to have global U(1) gauge anomaly. That is, the partition function of the projected theory picks up a phase under large U(1) gauge transformations. This anomalous phase is shown to be a minus sign, and hence leads to the $\mathbb{Z}_2$ classification. In the second example, by computing the global gravitational anomaly [114, 115] of the Majorana surface states of class D+R$_+$ TSCs, we study the "collapse" of non-interacting classification. The resulting projected theories are then shown to be anomalous under large diffeomorphisms (coordinate transformations).

In a similar vein, in Ref. [116], the (3+1)d Weyl fermion on the surface of the (4+1)d quantum Hall system is shown to fail to be modular invariant in the presence of a background U(1) gauge field.

The rest of this chapter is organized as follows. In the remaining part of this section, we introduce some notations that will be used in the main text. In Sec. 3.2, we establish the gauge and diffeomorphism invariance of the (2+1)d Dirac fermion theory defined on a spacetime three-torus, following Refs. [117, 118]. In Sec. 3.3, we study (3+1)d TIs protected by electromagnetic U(1) and CP symmetries. The surface theory projected by symmetries is shown to be anomalous, as its (projected) partition function is not invariant under large U(1) gauge transformations, but picks up a minus sign, characterizing the $\mathbb{Z}_2$ classification of the bulk phase. (3+1)d TSCs protected by reflection symmetry are studied in Sec. 3.4, where we discuss the invariance/non-invariance of the surface partition function, defined on the three-torus and its descendants generated by the orientifold projection, under large diffeomorphisms. We then conclude in Sec. 3.5.

Notations

The partition functions of the (2+1)d surface theories discussed in the text can be represented in terms of partition functions of (1+1)d theories. Here, we summarize the properties of these (1+1)d partition functions.

[2] While we are not to be restricted to relativistic systems in condensed matter physics, some universal physical properties of general, non-relativistic systems in the long wavelength limit, such as the band topology or the electromagnetic responses, are often encoded in topological field theories. Since topological, these theories respect the Lorentz symmetry, which guarantees the CPT invariance. In addition, from the perspective of topological classification of states of matter, classifying SPT phases of non-interacting fermion systems, for example, can be done solely in terms of Dirac operators with symmetry restrictions. Since a Dirac Hamiltonian has a CPT invariant form, we expect to obtain the same classification for all CPT equivalent systems, e.g., CP-protected TIs to class AII TIs, and classes D+R$_+$ TSCs to class DIII TSCs discussed here.

The partition function of a (1+1)d chiral fermion (Weyl fermion) on the two-torus T^2 with the modular parameter $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$, in the presence of spatial U(1) flux a and temporal U(1) flux b, is defined as [54, 96]

$$\begin{aligned} A^R_{[a,b]}(\tau) &\equiv \frac{1}{\eta(\tau)}\vartheta\begin{bmatrix} a-1/2 \\ b-1/2 \end{bmatrix}(0,\tau), \\ A^L_{[a,b]}(\tau) &= \left(A^R_{[a,b]}(\tau)\right)^*, \end{aligned} \tag{3.1}$$

where $\eta(\tau)$ is the Dedekind eta function and $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](v,\tau)$ is the theta function with characteristics. $A^R_{[a,b]}(\tau)$ has the following properties:

$$\begin{aligned} A^R_{[a,b]}(\tau) &= A^R_{[a+1,b]}(\tau) = e^{-2\pi i(a-1/2)}A^R_{[a,b+1]}(\tau), \\ A^R_{[a,b]}(\tau+1) &= e^{-\pi i(a^2-1/6)}A^R_{[a,b+a]}(\tau), \\ A^R_{[a,b]}\left(-1/\tau\right) &= e^{-2\pi i(-a+1/2)(b-1/2)}A^R_{[-b,a]}(\tau). \end{aligned} \tag{3.2}$$

The partition function of a (1+1)d massive Dirac fermion on T^2 with twisted boundary conditions (fluxes a and b) is given by the "massive theta function" $\Theta_{[a,b]}(\tau;m)$ [119, 120]:

$$\Theta_{[a,b]}(\tau;m) \equiv e^{4\pi\tau_2\Delta(m;a)}\prod_{s\in\mathbb{Z}+a}\left|1-e^{-2\pi\tau_2\sqrt{m^2+s^2}+2\pi i\tau_1 s+2\pi i b}\right|^2, \tag{3.3}$$

where $\Delta(m;a)$ is the regularized zero-point energy:

$$\Delta(m;a) \equiv \frac{1}{2}\sum_{s\in\mathbb{Z}+a}\sqrt{m^2+s^2} - \frac{1}{2}\int_{-\infty}^{\infty}dk\sqrt{m^2+k^2} = -\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\int_0^{\infty}dte^{-tn^2-\frac{\pi^2m^2}{t}}\cos(2\pi n a_x). \tag{3.4}$$

The massive theta function $\Theta_{[a,b]}(\tau;m)$ has the following properties:

$$\begin{aligned} &\Theta_{[a,b]}(\tau;m) = \Theta_{[-a,-b]}(\tau;m) = \Theta_{[a+r,b+s]}(\tau;m), \quad r,s\in\mathbb{Z}, \\ &\Theta_{[a,b]}(\tau+1;m) = \Theta_{[a,b+a]}(\tau;m), \\ &\Theta_{[a,b]}\left(-1/\tau;m|\tau|\right) = \Theta_{[b,-a]}(\tau;m), \\ &\lim_{m\to 0}\Theta_{[a,b]}(\tau;m) = \left|A^R_{[a,b]}(\tau)\right|^2 = \left|A^L_{[a,b]}(\tau)\right|^2. \end{aligned} \tag{3.5}$$

3.2 Large U(1) gauge and diffeomorphism invariance of (2+1)d fermion theory

In this section, we quantize the (2+1)d free Dirac fermion theory on a flat spacetime three-torus T^3, in the presence of background U(1) gauge field and metric. The invariance of the partition function under large U(1) gauge transformations and 3d modular transformations $\mathrm{SL}(3,\mathbb{Z})$, the mapping class group of T^3, will be established. A discussion for the 2d modular invariance of the Dirac fermion theory on two torus T^2, as a warm up, is reviewed in Appendix B.1.

We closely follow the analysis and notations in Ref. [117]. (See also Refs. [118] for related works.) In Ref. [117], the partition function of a chiral self-dual two-form gauge field on a 6d spacetime torus T^6, and its invariance under $\mathrm{SL}(6,\mathbb{Z})$, the mapping class group of the six-torus, was studied. In Ref. [117], the theory is quantized (regularized) in a way manifestly symmetric under $\mathrm{SL}(5,\mathbb{Z})$. It was then shown that the partition function has an additional $\mathrm{SL}(2,\mathbb{Z})$ invariance, and together with the $\mathrm{SL}(5,\mathbb{Z})$ invariance, the full $\mathrm{SL}(6,\mathbb{Z})$ invariance was proven. By properly adopting this strategy, we show the $\mathrm{SL}(3,\mathbb{Z})$ invariance and the large gauge invariance of the (2+1)d Dirac fermion theory.

While our focus in this section is on the complex or Dirac fermion, the case for real or Majorana fermions can be studied in a similar way. The modular properties studied here are expected to be straightforwardly generalized to higher dimensions, e.g., $\mathrm{SL}(n,\mathbb{Z})$ invariance for the partition function.

The Dirac fermion theory on three torus T^3

Background metric

A flat three-torus is parameterized by five "modular parameters", $R_{1,2}/R_0$, α, β, and γ, where R_μ are the radii for the μ-th directions, and α,β,γ and related to the angles between directions 0 and 1, 1 and 2, and 0 and 2, respectively. The dreibein is given by $(\mu, A = 0,1,2)$

$$e^A{}_\mu = \begin{pmatrix} R_0 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\gamma & -\beta & 1 \end{pmatrix} = \begin{pmatrix} R_0 & 0 & 0 \\ -\alpha R_1 & R_1 & 0 \\ -\gamma R_2 & -\beta R_2 & R_2 \end{pmatrix}, \tag{3.6}$$

and its inverse is given by

$$e^{\star}_{A}{}^{\mu} = \begin{pmatrix} \frac{1}{R_0} & \frac{\alpha}{R_0} & \frac{\alpha\beta+\gamma}{R_0} \\ 0 & \frac{1}{R_1} & \frac{\beta}{R_1} \\ 0 & 0 & \frac{1}{R_2} \end{pmatrix}, \tag{3.7}$$

such that $e^{A}{}_{\mu} e^{\star}_{A}{}^{\nu} = \delta_{\mu}{}^{\nu}$ and $e^{A}{}_{\mu} e^{\star}_{B}{}^{\mu} = \delta^{A}{}_{B}$. The Euclidean metric is

$$g_{\mu\nu} = e^{A}{}_{\mu} e^{B}{}_{\nu} \delta_{AB} = \begin{pmatrix} R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & -\alpha R_1^2 + \beta\gamma R_2^2 & -\gamma R_2^2 \\ -\alpha R_1^2 + \beta\gamma R_2^2 & R_1^2 + \beta^2 R_2^2 & -\beta R_2^2 \\ -\gamma R_2^2 & -\beta R_2^2 & R_2^2 \end{pmatrix}, \tag{3.8}$$

and the line element is given by $ds^2 = g_{\mu\nu} d\theta^{\mu} d\theta^{\nu}$, where $0 \le \theta^{\mu} \le 2\pi$ are angular variables.

The group $\mathrm{SL}(3,\mathbb{Z})$ is generated by two modular transformations [121]:

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

The dreiben and metric are transformed as

$$\begin{aligned} e^{A}{}_{\mu} &\xrightarrow{L} \left(eL^T\right)^{A}{}_{\mu} = L_{\mu}{}^{\rho} e^{A}{}_{\rho}, \\ e^{\star}_{A}{}^{\mu} &\xrightarrow{L} \left(e^{\star}L^{-1}\right)_{A}{}^{\mu} = e^{\star}_{A}{}^{\rho} \left(L^{-1}\right)_{\rho}{}^{\mu}, \\ g_{\mu\nu} &\xrightarrow{L} \left(LgL^T\right)_{\mu\nu} = L_{\mu}{}^{\rho} L_{\nu}{}^{\sigma} g_{\rho\sigma}, \end{aligned} \tag{3.10}$$

for any $\mathrm{SL}(3,\mathbb{Z})$ elements $L = U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$. In particular,

$$g_{\mu\nu} \xrightarrow{U_2} (U_2 g U_2^T)_{\mu\nu} = \begin{pmatrix} R_0^2 + (\alpha-1)^2 R_1^2 + (\gamma+\beta)^2 R_2^2 & -(\alpha-1) R_1^2 + \beta(\gamma+\beta) R_2^2 & -(\gamma+\beta) R_2^2 \\ -(\alpha-1) R_1^2 + \beta(\gamma+\beta) R_2^2 & R_1^2 + \beta^2 R_2^2 & -\beta R_2^2 \\ -(\gamma+\beta) R_2^2 & -\beta R_2^2 & R_2^2 \end{pmatrix}, \tag{3.11}$$

which corresponds to the changes

$$\alpha \to \alpha - 1, \quad \gamma \to \gamma + \beta \quad \text{(while } R_0,\ R_1,\ R_2, \text{ and } \beta \text{ are unchanged).} \tag{3.12}$$

The less trivial generator U_1 can be further decomposed into two transformations as

$$U_1 = U_1' M, \quad U_1' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.13}$$

The transformation U_1' acts on the metric as

$$g_{\mu\nu} \xrightarrow{U_1'} (U_1' g U_1'^T)_{\mu\nu} = \begin{pmatrix} R_1^2 + \beta^2 R_2^2 & \alpha R_1^2 - \beta\gamma R_2^2 & \beta R_2^2 \\ \alpha R_1^2 - \beta\gamma R_2^2 & R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & -\gamma R_2^2 \\ \beta R_2^2 & -\gamma R_2^2 & R_2^2 \end{pmatrix}, \tag{3.14}$$

which corresponds to the changes

$$R_0 \to R_0/|\tau_{2d}|, \quad R_1 \to R_1|\tau_{2d}|, \quad \alpha \to -\alpha/|\tau_{2d}|^2 \quad (\text{or } \tau_{2d} \to -1/\tau_{2d}), \quad \gamma \to -\beta, \quad \beta \to \gamma \quad (\text{while } R_2 \text{ is unchanged}), \tag{3.15}$$

with

$$\tau_{2d} \equiv \alpha + i r_{01}, \quad r_{\mu\nu} \equiv R_\mu / R_\nu, \tag{3.16}$$

while M acts on the metric as

$$g_{\mu\nu} \xrightarrow{M} (M g M^T)_{\mu\nu} = \begin{pmatrix} R_0^2 + \alpha^2 R_1^2 + \gamma^2 R_2^2 & \gamma R_2^2 & -\alpha R_1^2 + \beta\gamma R_2^2 \\ \gamma R_2^2 & R_2^2 & \beta R_2^2 \\ -\alpha R_1^2 + \beta\gamma R_2^2 & \beta R_2^2 & R_1^2 + \beta^2 R_2^2 \end{pmatrix}. \tag{3.17}$$

The Euclidean action for the Dirac fermion on T^3 is then given by

$$\begin{aligned} S_E &= \frac{1}{(2\pi)^2} \int d^3\theta \, (\det e) \, \bar{\psi} \left(\Gamma^A e^{\star}{}_A{}^{\mu} \frac{\partial}{\partial \theta^\mu} \right) \psi \\ &= \frac{1}{(2\pi)^2} \int_0^{2\pi R_0} d\tau \int_0^{2\pi R_1} dx \int_0^{2\pi R_2} dy \, \bar{\psi} \left[\Gamma^0 \partial_\tau + \alpha \frac{R_1}{R_0} \Gamma^0 \partial_x + (\alpha\beta + \gamma) \frac{R_2}{R_0} \Gamma^0 \partial_y + \Gamma^1 \partial_x + \beta \frac{R_2}{R_1} \Gamma^1 \partial_y + \Gamma^2 \partial_y \right] \psi, \end{aligned} \tag{3.18}$$

where ψ is the two-component Dirac field, $\bar{\psi} = \psi^\dagger \Gamma^0$, $\tau = R_0 \theta^0$, $x = R_1 \theta^1$, $y = R_2 \theta^2$, and the gamma matrices Γ^A satisfy $\{\Gamma^A, \Gamma^B\} = 2\delta^{AB}$.

Background flux

In addition to the background metric, we also introduce the background U(1) gauge field (flux) on T^3 to twist the boundary conditions of the Dirac fermion theory. More specifically, in the path integral language, we consider the boundary conditions

$$\begin{aligned}
&\psi(\tau, x+2\pi R_1, y) = e^{2\pi i a_x}\psi(\tau,x,y),\\
&\psi(\tau, x, y+2\pi R_2) = e^{2\pi i a_y}\psi(\tau,x,y),\\
&\psi(\tau+2\pi R_0, x-2\pi\alpha R_1, y-2\pi(\alpha\beta+\gamma)R_2) = e^{2\pi i a_\tau}\psi(\tau,x,y),
\end{aligned} \tag{3.19}$$

where $(a_\tau, a_x, a_y) \equiv \mathsf{a}$ represents the background U(1) gauge field twisting the boundary conditions.

Partition function

We now quantize the (2+1)d theory and compute the properly regularized partition function, denoted by $Z_{[\mathsf{a}]}(g)$, which depends on the background flux a and metric g. The partition function can be evaluated by the path integral on T^3, $Z_{[\mathsf{a}]}(g) = \int \mathcal{D}[\psi^\dagger,\psi]\exp(-S_E)$, with ψ satisfying the twisted boundary conditions (3.19), or alternatively, in the operator language, by the trace

$$Z_{[\mathsf{a}]}(g) = \mathrm{Tr}_{a_x a_y}\left[e^{2\pi i(a_\tau - 1/2)F} e^{-2\pi R_0 H'}\right], \tag{3.20}$$

where H' is the "boosted" Hamiltonian (in the presence of non-vanishing angles α, β, and γ) obtained from S_E and given by

$$H' = H - i\alpha\frac{R_1}{R_0}P_x - i(\alpha\beta+\gamma)\frac{R_2}{R_0}P_y, \tag{3.21}$$

with

$$\begin{aligned}
H &= \frac{1}{(2\pi)^2}\int dxdy\,\bar\psi\left(\Gamma^1\partial_x + \beta\frac{R_2}{R_1}\Gamma^1\partial_y + \Gamma^2\partial_y\right)\psi,\\
P_i &= \frac{1}{(2\pi)^2}\int dxdy\,\psi^\dagger(-i\partial_i\psi), \quad i = x,y,
\end{aligned} \tag{3.22}$$

being the Hamiltonian and momenta. Tr_{a_x,a_y} means the trace is taken over the Fock space of the fermion theory for the spatial boundary conditions specified by a_x and a_y. The twisted boundary condition in the τ-direction is implemented by an operator insertion $\exp[2\pi i(a_\tau - 1/2)F]$, where F is the fermion number operator.

The fermion field operator satisfies the canonical anticommutation relation

$$\{\psi_\alpha(\mathsf{r}), \psi_\beta^\dagger(\mathsf{r}')\} = (2\pi)^2 \delta_{\alpha\beta} \sum_{m_1, m_2 \in \mathbb{Z}} \delta(x - x' + 2\pi m_1 R_1)\delta(y - y' + 2\pi m_2 R_2), \tag{3.23}$$

where $\mathsf{r} = (x, y)$ and α, β are spinor indices. The trace can be evaluated explicitly by the Fourier mode expansion of the fermion field operator. With the twisted boundary conditions, the fermion field operator is expanded as

$$\psi(\mathsf{r}) = \frac{1}{\sqrt{R_1 R_2}} \sum_{s_x \in \mathbb{Z}+a_x} \sum_{s_y \in \mathbb{Z}+a_y} e^{ix\frac{s_x}{R_1} + iy\frac{s_y}{R_2}} \tilde{\psi}(\mathsf{s}), \tag{3.24}$$

where $\mathsf{s} = (s_x, s_y)$ and

$$\{\tilde{\psi}_\alpha(\mathsf{s}), \tilde{\psi}_\beta^\dagger(\mathsf{s}')\} = \delta_{\alpha\beta}\delta_{\mathsf{s}\mathsf{s}'}. \tag{3.25}$$

Correspondingly, the Hamiltonian can be expanded as

$$\begin{aligned} H &= \sum_{s_x \in \mathbb{Z}+a_x} \sum_{s_y \in \mathbb{Z}+a_y} \tilde{\psi}^\dagger(\mathsf{s}) \mathcal{H}(\mathsf{s}) \tilde{\psi}(\mathsf{s}), \\ \mathcal{H}(\mathsf{s}) &= \Gamma^0 \left[\Gamma^1 \frac{i s_x}{R_1} + \beta \frac{R_2}{R_1} \Gamma^1 \frac{i s_y}{R_2} + \Gamma^2 \frac{i s_y}{R_2} \right]. \end{aligned} \tag{3.26}$$

The single-particle Hamiltonian $\mathcal{H}(\mathsf{s})$ can be diagonalized with eigenvectors $\vec{u}_\pm(\mathsf{s})$ and eigenvalues $\pm\varepsilon(\mathsf{s})$:

$$\begin{aligned} &\mathcal{H}(\mathsf{s})\vec{u}_\pm(\mathsf{s}) = \pm\varepsilon(\mathsf{s})\vec{u}_\pm(\mathsf{s}), \\ &\varepsilon(\mathsf{s}) = \sqrt{g_2^{ij} s_i s_j} = \sqrt{\left(\frac{s_x}{R_1} + \beta\frac{s_y}{R_1}\right)^2 + \left(\frac{s_y}{R_2}\right)^2}, \end{aligned} \tag{3.27}$$

where

$$g_2^{ij} \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{R_1^2} & \frac{\beta}{R_1^2} \\ \frac{\beta}{R_1^2} & \frac{\beta^2}{R_1^2} + \frac{1}{R_2^2} \end{pmatrix}. \tag{3.28}$$

The Hamiltonian can be diagonalized by the eigen basis $\chi(\mathsf{s}) := [\chi_+(\mathsf{s}), \chi_-(\mathsf{s})]^T$, which are related to the original fermion operators $\tilde{\psi}(\mathsf{s})$ as

$$\begin{bmatrix} \tilde{\psi}_1(\mathsf{s}) \\ \tilde{\psi}_2(\mathsf{s}) \end{bmatrix} = \begin{bmatrix} u_{1+}(\mathsf{s}) & u_{1-}(\mathsf{s}) \\ u_{2+}(\mathsf{s}) & u_{2-}(\mathsf{s}) \end{bmatrix} \begin{bmatrix} \chi_+(\mathsf{s}) \\ \chi_-(\mathsf{s}) \end{bmatrix},$$
$$\begin{bmatrix} \chi_+(\mathsf{s}) \\ \chi_-(\mathsf{s}) \end{bmatrix} = \begin{bmatrix} u^*_{1+}(\mathsf{s}) & u^*_{2+}(\mathsf{s}) \\ u^*_{1-}(\mathsf{s}) & u^*_{2-}(\mathsf{s}) \end{bmatrix} \begin{bmatrix} \tilde{\psi}_1(\mathsf{s}) \\ \tilde{\psi}_2(\mathsf{s}) \end{bmatrix}. \tag{3.29}$$

The Hamiltonian in the eigen basis is given by

$$\begin{aligned} H &= \sum_{\mathsf{s}} \varepsilon(\mathsf{s}) \left[\chi^\dagger_+(\mathsf{s})\chi_+(\mathsf{s}) - \chi^\dagger_-(\mathsf{s})\chi_-(\mathsf{s})\right] \\ &= \sum_{\mathsf{s}} \varepsilon(\mathsf{s}) \left[\chi^\dagger_+(\mathsf{s})\chi_+(\mathsf{s}) + \chi_-(x)\chi^\dagger_-(\mathsf{s})\right] - \sum_{\mathsf{s}} \varepsilon(\mathsf{s}) \\ &=: H : + E_{\mathrm{GS}} \end{aligned} \tag{3.30}$$

where $:\cdots:$ is the normal ordering with respect to the Fock vacuum and $E_{\mathrm{GS}} = -\sum_{\mathsf{s}} \varepsilon(\mathsf{s})$ is the ground-state energy.

The ground-state energy needs to be properly regularized. As shown in Appendix B.2, we have

$$E_{\mathrm{GS}[\mathsf{a}]}(g) = -\sum_{\mathsf{s}} \sqrt{g_2^{ij} s_i s_j} = \frac{1}{4\pi^2} \sqrt{\det(g_{2ij})} \sum_{\mathsf{n} \neq 0 \in \mathbb{Z}^2} \frac{\cos\left(2\pi i a^i n_i\right)}{\left(g_2^{ij} n_i n_j\right)^{\frac{3}{2}}}, \tag{3.31}$$

where $s_{x,y}$ are separated into their integral and fractional parts as

$$s_i = n_i + a_i, \quad n_i \in \mathbb{Z}, \quad i = x, y. \tag{3.32}$$

Similarly, the ground-state momentum and the fermion number can be regularized as

$$\begin{aligned} (P_i)_{\mathrm{GS}} &= \sum_{s_i} s_i = \sum_{s_i > 0} s_i + \sum_{s_i < 0} s_i = \zeta(-1, a_i) - \zeta(-1, 1 - a_i) = 0, \\ F_{\mathrm{GS}} &= \sum_{\mathsf{s}} 1 - \sum_{\mathsf{s}} 1 = 0, \end{aligned} \tag{3.33}$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s}$ is the Hurwitz zeta function defined by analytic continuation from the region $\mathrm{Re}(s) > 1$ and we have used $\zeta(-1, x) = \frac{1}{24} - \frac{1}{2}\left(x - \frac{1}{2}\right)^2$.

With the regularization, the partition function (with boundary conditions twisted by a_x and a_y), given by the trace in (3.20), is evaluated as

$$Z_{[\mathsf{a}]}(g) = e^{-2\pi R_0 E_{\mathrm{GS}}} \prod_{s_y \in \mathbb{Z}+a_y} \prod_{s_x \in \mathbb{Z}+a_x} \left| 1 - e^{-2\pi R_0 \varepsilon(\mathsf{s}) + 2\pi i \alpha s_x + 2\pi i(\alpha\beta+\gamma)s_y + 2\pi i a_\tau} \right|^2, \tag{3.34}$$

which can also be expressed as the infinite product of massive theta functions defined in (3.3):

$$Z_{[\mathsf{a}]}(g) = \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[a_x+\beta s_y, a_\tau+\gamma s_y]}\left(\tau_{2d}; r_{12} s_y\right). \tag{3.35}$$

We now demonstrate the invariance of the partition function under large U(1) gauge transformations and modular transformations.

Large U(1) gauge invariance of the partition function

We first check the invariance of the partition function under large U(1) gauge transformations $a_{x,y,\tau} \to a_{x,y,\tau} + 1$. The invariance under $a_{x,\tau} \to a_{x,\tau} + 1$ is obvious from Eq. (3.35), using the properties of the massive theta function listed in (3.5). To check the invariance of the partition function under $a_y \to a_y + 1$, we note that this amounts to a simple shift $s_y \to s_y + 1$ in the infinite product in Eq. (3.35). To sum up, we conclude the large U(1) gauge invariance of the partition function.

Modular invariance of the partition function

By using the expressions (3.34) and (3.35) of the partition function, we can show that $Z_{[\mathsf{a}]}(g)$ has the following property:

$$Z_{[L\mathsf{a}]}(LgL^T) = Z_{[\mathsf{a}]}(g), \quad \text{or} \quad Z_{[\mathsf{a}]}(LgL^T) = Z_{[L^{-1}\mathsf{a}]}(g), \tag{3.36}$$

where $L \in \mathrm{SL}(3,\mathbb{Z})$. This means the Dirac fermion, when coupled to both background U(1) gauge field and metric, is anomaly-free under any large diffeomorphisms (together with the induced gauge transformations) on T^3. The claim (3.36) can be shown by checking how $Z_{[\mathsf{a}]}(g)$ transforms under $U_1 = U_1' M$ and U_2, defined in Eqs. (3.9) and (3.13). Here we leave the detail of the derivation to Appendix B.3.

We now show that the partition function, once projected by the fermion number parity only [i.e., in the absence of U(1) gauge field], is modular invariant. We consider the sum over all periodic/antiperiodic boundary conditions, which are specified by boundary conditions $\mathsf{a} = (0,0,0)$, $(\frac{1}{2},0,0)$, $\ldots$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, corresponding to the $2^3 = 8$ spin structures when defining fermions (spinors) on T^3. The resulting total partition function is given by

$$\mathcal{Z}^{tot}(g) = \sum_{a_{x,y,\tau}=0,1/2} \epsilon_{\mathsf{a}} Z_{[\mathsf{a}]}(g), \tag{3.37}$$

where ϵ_{a} are weights ("discrete torsion") assigned to different sectors with partition functions twisted by a. From Eq. (3.36), we see that by choosing $\epsilon_{\mathsf{a}} = 1$ for all a, the total partition function is modular invariant:

$$\mathcal{Z}^{tot}(LgL^T) = \mathcal{Z}^{tot}(g),\ L \in \mathrm{SL}(3, \mathbb{Z}). \tag{3.38}$$

3.3 Surface theory of (3+1)d CP symmetric topological insulators

Based on the result from the previous section, now we can compute quantum anomalies of an anomalous surface theory and interpret them as a signal of the existence of the nontrivial bulk SPT phases. In this section, we identify a global U(1) gauge anomaly of the surface theory of (3+1)d CP (charge conjugation × reflection) symmetric TIs, which are related to, by CPT-theorem, (3+1)d time-reversal symmetric TIs.

3.3.1 Surface theory

Let us consider the following surface Dirac Hamiltonian of (3+1)d TIs:

$$H = \frac{1}{(2\pi)^2} \int dxdy\, \psi^\dagger(\mathsf{r})(-i\sigma_2\partial_x - i\sigma_1\partial_y)\psi(\mathsf{r}), \tag{3.39}$$

where $\sigma_{1,2,3}$ are the Pauli matrices, the spatial coordinate $\mathsf{r} = (x, y) \in [0, 2\pi R_1) \times [0, 2\pi R_2)$ parameterizes the 2d surface, and $\psi(\mathsf{r}) = [\psi_1(\mathsf{r}), \psi_2(\mathsf{r})]^T$ and $\psi^\dagger(\mathsf{r}) = [\psi_1^\dagger(\mathsf{r}), \psi_2^\dagger(\mathsf{r})]$ are two-component fermion annihilation and creation operators, respectively. The Hamiltonian is invariant under the following time-reversal symmetry:

$$\mathscr{T}\psi(\mathsf{r})\mathscr{T}^{-1} = i\sigma_2\psi(\mathsf{r}), \quad \mathscr{T}^2 = (-1)^F, \tag{3.40}$$

where F is the fermion number operator. This time-reversal symmetry prohibits the mass term $\psi^\dagger\sigma_3\psi$ since $\mathscr{T}\psi^\dagger\sigma_3\psi\mathscr{T}^{-1} = -\psi^\dagger\sigma_3\psi$.

Alternatively, in the following, we will take the Dirac Hamiltonian (3.39) as a surface theory of bulk CP symmetric TIs. By CPT-theorem, the classification of CP symmetric TIs are expected to be the same as the classification of time-reversal symmetric TIs. That is, CP symmetric insulators in (3+1)d are also classified by a $\mathbb{Z}_2$ topological number [111]. Within the surface theory, the action of CP symmetry is given by

$$(\mathscr{C}\mathscr{P})\psi(x,y)(\mathscr{C}\mathscr{P})^{-1} = \sigma_3[\psi^\dagger(x, 2\pi R_2 - y)]^T,$$
$$(\mathscr{C}\mathscr{P})\psi^\dagger(x,y)(\mathscr{C}\mathscr{P})^{-1} = \psi(x, 2\pi R_2 - y)^T\sigma_3,$$
$$(\mathscr{C}\mathscr{P})^2 = 1. \tag{3.41}$$

[This is the only CP symmetry of the Dirac kinetic term $\mathcal{H}(k_x,k_y) = k_x\sigma_2 + k_y\sigma_1$ since $\sigma_3\mathcal{H}^T(-k_x,k_y)\sigma_3^{-1} = -\mathcal{H}(k_x,k_y)$.] Fermion bilinears in the surface theory are transformed as $\psi^\dagger M\psi \to \psi^T U^\dagger M U(\psi^\dagger)^T = -\psi^\dagger U^T M^T U^*\psi$. In particular, the mass is odd under CP; the surface theory, at least at quadratic level, cannot be gapped without breaking symmetries, U(1) $\rtimes$ CP. On the other hand, a CP preserving mass exists if we double this theory (or more generally if the number of the Dirac fermions is even), and the corresponding surface theory can be gapped. Since massive fermions can always be regularized to construct a well-defined quantum theory, an even number of the surface fermions (3.39) is always anomaly-free (while preserving the symmetries). However, an odd number of the surface fermions may suffer from anomalies. In the following, we will identify a quantum anomaly of the surface theory (3.39) under large U(1) gauge transformation when CP symmetry is strictly enforced.

3.3.2 Projected partition function by CP symmetry

We now consider CP projection of the surface theory (3.39) and ask if the projected theory is still invariant under large gauge transformations (as we have seen in the theory without symmetry projection). This leads to formulating the fermion theory on unorientable spacetime manifolds such as $S^1 \times K$, where K is the Klein bottle. As our main focus here is on large gauge transformations but not on modular transformations, the parameters α, β, and γ are set to zero in the following discussion. Also, the twisted boundary conditions by U(1) are consistent with CP symmetry only when $a_{\tau,x} = 0, 1/2$, while a_y is not constrained by CP symmetry. We will thus study the large gauge transformation $a_y \to a_y + 1$.

The partition function of the surface theory projected by CP is given by the trace

$$\mathrm{Tr}_{a_x,a_y}\left[\frac{1}{2}(1+\mathscr{C}\mathscr{P})e^{2\pi i(a_\tau - 1/2)F}e^{-2\pi R_0 H}\right]. \tag{3.42}$$

Upon projection by CP, we will focus on the CP symmetric boundary conditions $a_{\tau,x} = 0, 1/2$ and $a_y \in [0,1)$. The first term in the projected trace, which is invariant under $a_y \to a_y + 1$, is already discussed in Sec. 3.2. Our focus below will be the second term in the projected trace, which we call the CP twisted partition function:

$$Z^{\mathrm{CP}}_{[\mathsf{a}]} = \mathrm{Tr}_{a_x,a_y}\, \mathscr{C}\mathscr{P} e^{2\pi i(a_\tau - 1/2)F} e^{-2\pi R_0 H}. \tag{3.43}$$

Because of CP symmetry, from the eigenvectors $\vec{u}_\pm$ at s, we can construct eigenvectors at $\bar{\mathsf{s}} = (-s_x, s_y)$:

$$\mathcal{H}(\bar{\mathsf{s}})\sigma_3 \vec{u}^*_\pm(\mathsf{s}) = \mp\varepsilon(\mathsf{s})\sigma_3\vec{u}^*_\pm(\mathsf{s}). \tag{3.44}$$

The CP operator acts on the Fourier components of the original fermion operators as

$$(\mathscr{C}\mathscr{P})\tilde{\psi}(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} = \sigma_3 \left[\tilde{\psi}^\dagger(\bar{\mathsf{s}})\right]^T. \tag{3.45}$$

On the other hand, the CP action on the eigen basis $\chi^\dagger_\pm, \chi_\pm$ is deduced as

$$(\mathscr{C}\mathscr{P})\chi(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} = \begin{bmatrix} \langle u_+(\mathsf{s})|\sigma_3 K|u_+(\bar{\mathsf{s}})\rangle & \langle u_+(\mathsf{s})|\sigma_3 K|u_-(\bar{\mathsf{s}})\rangle \\ \langle u_-(\mathsf{s})|\sigma_3 K|u_+(\bar{\mathsf{s}})\rangle & \langle u_-(\mathsf{s})|\sigma_3 K|u_-(\bar{\mathsf{s}})\rangle \end{bmatrix} \left[\chi^\dagger(\bar{\mathsf{s}})\right]^T, \tag{3.46}$$

where K is the complex conjugation operator, and $\langle u_\pm(\mathsf{s})|\sigma_3 K|u_\pm(\bar{\mathsf{s}})\rangle = \vec{u}^*_\pm(\mathsf{s}) \cdot \sigma_3 \vec{u}^*_\pm(\bar{\mathsf{s}})$, etc. Since $\vec{u}_\pm(\mathsf{s})$ and $\sigma_3\vec{u}^*_\pm(\bar{\mathsf{s}})$ are both eigenvectors of Hamiltonian $\mathcal{H}(\mathsf{s})$ but with different energies, their overlap should be zero. Therefore,

$$\begin{aligned} (\mathscr{C}\mathscr{P})\chi_+(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} &= \langle u_+(\mathsf{s})|\sigma_3 K|u_-(\bar{\mathsf{s}})\rangle \chi^\dagger_-(\bar{\mathsf{s}}), \\ (\mathscr{C}\mathscr{P})\chi_-(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} &= \langle u_-(\mathsf{s})|\sigma_3 K|u_+(\bar{\mathsf{s}})\rangle \chi^\dagger_+(\bar{\mathsf{s}}). \end{aligned} \tag{3.47}$$

The transformation law of $\chi_\pm$ under CP depends on the choice of eigen functions $\vec{u}_\pm$. A choice for the eigenvectors is

$$|u_\pm(\mathsf{s})\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm\Delta^*/|\Delta| \\ 1 \end{bmatrix}, \quad \Delta = s_y + i s_x. \tag{3.48}$$

For this choice of eigenfunctions,

$$\langle u_+(\mathsf{s})|\sigma_3 K|u_-(\bar{\mathsf{s}})\rangle = \langle u_-(\mathsf{s})|\sigma_3 K|u_+(\bar{\mathsf{s}})\rangle = -1. \tag{3.49}$$

As we can choose the phase of the eigenvectors freely,

$$|u_{\pm}(\mathsf{s})\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm\Delta/|\Delta| \end{bmatrix} \tag{3.50}$$

is also an eigenfunction. In this gauge,

$$\langle u_{+}(\mathsf{s})|\sigma_3 K|u_{-}(\bar{\mathsf{s}})\rangle = \langle u_{-}(\mathsf{s})|\sigma_3 K|u_{+}(\bar{\mathsf{s}})\rangle = 1. \tag{3.51}$$

In either choice, the result can be summarized as

$$\begin{aligned} (\mathscr{C}\mathscr{P})\chi_{+}(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} &= \eta_{+}\chi_{-}^{\dagger}(\bar{\mathsf{s}}), \\ (\mathscr{C}\mathscr{P})\chi_{-}(\mathsf{s})(\mathscr{C}\mathscr{P})^{-1} &= \eta_{-}\chi_{+}^{\dagger}(\bar{\mathsf{s}}), \end{aligned} \tag{3.52}$$

where $\eta_{\pm}$ is s-independent. The product $\eta := \eta_{+}\eta_{-} = 1$ is gauge invariant.

The CP twisted partition function $Z^{\mathrm{CP}}_{[\mathsf{a}]}$ can then be computed explicitly as

$$Z^{\mathrm{CP}}_{[\mathsf{a}]} = e^{-2\pi R_0 E_{\mathrm{GS}}} P_{[a_x,a_y]} \prod_{s_x \in \mathbb{Z}+a_x} \prod_{s_y \in \mathbb{Z}+a_y} \left[1 - e^{-4\pi R_0 \varepsilon(\mathsf{s})}\right], \tag{3.53}$$

where the prefactor $P_{[a_x,a_y]}$ is the CP eigenvalue of the ground state (the Fock vacuum). Note that the partition function does not depend on a_τ, which is projected out by CP. [3] In the following, we denote $Z^{\mathrm{CP}}_{[\mathsf{a}]} = Z^{\mathrm{CP}}_{[a_x,a_y]}$ and consider the two cases $a_x = 0, 1/2$ separately.

Periodic boundary condition in the x direction, $a_x = 0$: In this case, we can factor the twisted partition function into the product of 2d massless ($s_z = 0$) and massive ($s_x \neq 0$) modes

$$\begin{aligned} Z^{\mathrm{CP}}_{[a_x=0,a_y]} = P_{[a_x=0,a_y]} e^{2\pi R_0 \sum_{s_y} \varepsilon(s_x=0,s_y)} \prod_{s_y \in \mathbb{Z}+a_y} \left[1 - e^{-4\pi R_0 \varepsilon(s_x=0,s_y)}\right] \\ \times \prod_{s_x \in \mathbb{Z}^+} \left[e^{4\pi R_0 \sum_{s_y} \varepsilon(\mathsf{s})} \prod_{s_y \in \mathbb{Z}+a_y} \left|1 - e^{-4\pi R_0 \varepsilon(\mathsf{s})}\right|^2 \right], \end{aligned}$$

[3]When evaluating the CP twisted partition function (3.43), only the simultaneous eigenstates of H, $\mathscr{C}\mathscr{P}$, and F contribute to the trace. Since the eigenstates of $\mathscr{C}\mathscr{P}$ are charge neutral, it means $e^{2\pi i(a_\tau - 1/2)F}$ acts as the identity operator inside the trace, and therefore a_τ does not show up in Z^{CP}.

which can be expressed (as the sum $\sum_{s_y} \varepsilon(\mathsf{s})$ is regularized) in terms of (1+1)d partition functions defined in Sec. 3.1:

$$Z^{\mathrm{CP}}_{[a_x=0,a_y]} = P_{[a_x=0,a_y]} e^{-\pi i(a_y-1/2)} A^R_{[a_y,0]}(2ir_{02}) \prod_{s_x\in\mathbb{Z}^+} \Theta_{[a_y,0]}(2ir_{02}; r_{21}s_x), \tag{3.54}$$

where $r_{\mu\nu} \equiv R_\mu/R_\nu$. When imposing CP symmetry, it is reasonable to assume the CP-eigenvalue does not change under $a_y \to a_y + 1$ (see the discussion in the last chapter), i.e., $P_{[a_x=0,a_y+1]} = P_{[a_x=0,a_y]}$. Under this assumption, we have

$$Z^{\mathrm{CP}}_{[a_x=0,a_y+1]} = -Z^{\mathrm{CP}}_{[a_x=0,a_y]}, \tag{3.55}$$

where we note both $A^R_{[a_y,0]}$ and $\Theta_{[a_y,0]}$ are invariant under $a_x \to a_x+1$. The anomalous minus sign under the large gauge transformation, which comes from the 2d massless modes ($s_x = 0$) but not the massive modes ($s_x \neq 0$), signals a $\mathbb{Z}_2$ topological classification: the CP projected theory can only be realized as the surface theory of a (3+1)d bulk CP symmetric TI, which is CPT-conjugate to a (3+1)d time-reversal symmetric TI [111].

Antiperiodic boundary condition in the x direction, $a_x = 1/2$: In this case, the twisted partition function is given by

$$Z^{\mathrm{CP}}_{[a_x=1/2,a_y]} = P_{[a_x=1/2,a_y]} \prod_{s_x\in\mathbb{Z}^+-1/2} \Theta_{[a_y,0]}(2ir_{02}; r_{21}s_x). \tag{3.56}$$

Observe that there are no 2d massless modes arising in the expression of $Z^{\mathrm{CP}}_{[a_x=1/2,a_y]}$ (while the product of all massive modes changes from $\prod_{s_y\in\mathbb{Z}^+}$ to $\prod_{s_y\in\mathbb{Z}^+-1/2}$). This partition function is anomaly-free under the large gauge transformation, i.e.,

$$Z^{\mathrm{CP}}_{[a_x=1/2,a_y+1]} = Z^{\mathrm{CP}}_{[a_x=1/2,a_y]}. \tag{3.57}$$

For arbitrary number N of the Dirac fermion flavors, the result is summarized as:

$$\left(Z^{\mathrm{CP}}_{[a_x,a_y+1]}\right)^N = (-1)^{2N(a_x-1/2)} \left(Z^{\mathrm{CP}}_{[a_x=0,a_y]}\right)^N. \tag{3.58}$$

The surface theory, as projected (or twisted) by CP, is anomaly-free if and only if $N = 0 \mod 2$. This characterizes the $\mathbb{Z}_2$ classification of the bulk SPT phase.

3.4 Surface theory of (3+1)d reflection symmetric crystalline topological superconductors

In this section, we identify a global gravitational anomaly of the surface theory of (3+1)d reflection symmetric crystalline TSCs, which are related to, by CPT-theorem, (3+1)d time-reversal symmetric TSCs. While the $\mathbb{Z}_2$-type (gauge) anomaly in the surface of CP TIs agrees with the non-interacting classification of the bulk phase, the (gravitational) anomaly in the surface of reflection symmetric TSCs, as we will discuss later, sees only the reduction of non-interacting classification, and hence can detect the effect of interactions (in the case that the bulk gap is not destroyed by the interactions).

3.4.1 Surface theory

At the quadratic level, time-reversal symmetric superconductors in symmetry class DIII are classified by an integer topological invariant, the 3d winding number ν [9]. The topological invariant counts the number of gapless surface Majorana cones. For example, the B-phase of ^{3}He is a TSC (superfluid) with $\nu = 1$, and hosts, when terminated by a surface, a surface Majorana cone, which can be modeled, at low energies, by the Hamiltonian

$$H = \frac{1}{(2\pi)^2} \int d^2\mathsf{r}\, \lambda^T(-i\sigma_3\partial_x - i\sigma_1\partial_y)\lambda, \tag{3.59}$$

where $\sigma_{1,2,3}$ are the Pauli matrices, the spatial coordinate $\mathsf{r} = (x, y) \in [0, 2\pi R_1) \times [0, 2\pi R_2)$ parameterizes the 2d surface, and $\lambda(\mathsf{r})$ is a two-component real fermionic field satisfying $\lambda^\dagger(\mathsf{r}) = \lambda(\mathsf{r})$. The surface Hamiltonian is invariant under time-reversal $\mathscr{T}$ defined by $\mathscr{T}\lambda(\mathsf{r})\mathscr{T}^{-1} = i\sigma_2\lambda(\mathsf{r})$, where $\mathscr{T}^2$ is equal to the fermion number parity $\mathscr{G}_f = (-1)^F = \frac{1}{(2\pi)^2}\int d^2\mathsf{r}\, \lambda^T\sigma_2\lambda$. For TSCs with $\nu = N_f$, the surface modes can be modeled by N_f copies of the above surface Hamiltonian.

While, at the quadratic level, , one can verify that, for an arbitrary integer $\nu = N_f$, surface Majorana cones are stable against perturbations the surface Majorana cones may be destabilized once interactions are included. A number of arguments, such as "vortex condensation approach", "symmetry-preserving surface topological order", "cobordism approach" and so on [48, 49, 41, 50, 51], show that the surface Majorana cones are unstable against interactions when $\nu = 0 \bmod 16$, reducing the non-interacting integer classification to $\mathbb{Z}_{16}$.

Here, instead of time-reversal symmetry, we consider its CPT-conjugate, reflection or parity symmetry, which acts on the Majorana field as

$$\mathscr{P}\lambda(x,y)\mathscr{P}^{-1} = \sigma_3\lambda(x, 2\pi R_2 - y), \quad \mathscr{P}^2 = 1. \tag{3.60}$$

Upon demanding the invariance under parity (3.60), the Majorana Hamiltonian (3.59) describes the surface of symmetry class D + R_+ crystalline TSCs, which are, at the quadratic level, classified by the integral mirror Chern number [75, 76, 112]. Based on CPT-theorem, we expect, upon the inclusion of interactions, the integer classification collapses down to $\mathbb{Z}_{16}$.

To see the stability of the gapless Majorana mode at the quadratic level, note that the mass $\lambda^T\sigma_2\lambda$ is odd under parity (3.60) and prohibited. It is also interesting to note that while the uniform mass is not allowed, one could consider $\int d^2\mathsf{r}\, m(\mathsf{r})\lambda^T\sigma_2\lambda$ with $m(x, 2\pi R_2 - y) = -m(x,y)$. This perturbation gaps out the most part of the surface, but not completely. At the fixed points of P symmetry, $y = 0$ and $y = \pi R_2$, it leaves gapless modes localized at the domain walls. Note that this is similar to the chiral mode localized at a mass domain wall on the surface of time-reversal symmetric TIs. The difference, however, is that in the present case, the mass domain wall, as a whole, preserves the reflection symmetry, while the domain wall on the surface of TIs breaks time-reversal symmetry, except at the domain wall. The gapless mode at the domain wall consists of the N_f copies of Majorana fermions propagating in either $+x$ or $-x$ directions, depending on the overall sign of the mass domain wall for each flavor. For even N_f, we can always choose (technically) a set of mass parameters such that the gapless modes at the domain wall are made nonchiral (*e.g.*, by choosing different signs of the masses for different flavors of Majorana fermions). In this case, reflection symmetry acts on the (1+1)d gapless domain-wall fermions as an unitary on-site Z_2 symmetry. Using the result in Ref. [28], it can be shown that such gapless domain-wall states can be gapped without breaking the symmetry if the number of the non-chiral states is 0 mod 8. This means, when $N_f = 0$ mod 16, we can gapped out the surface of the crystalline TSCs while preserving the reflection symmetry at the same time. This gives the $\mathbb{Z}_{16}$ classification, as expected to the same as the case of class DIII TCSs, of the class D + R+ crystalline TSCs, upon the inclusion of interactions. A similar argument for the Z_8 classification of interacting crystalline TIs protected by reflection symmetry can be found in Ref. [122].

3.4.2 Projected partition function by reflection/parity symmetry

We now study the presence/absence of (global) gravitational anomalies of the surface theory (3.59), which signals the existence of the nontrivial bulk SPT phases. For convenience, we double the degrees of freedom and consider Dirac instead of Majorana fermion fields. (This is purely a matter of convenience. The analysis below can be repeated without referring to the Dirac fermion, and can be done solely in terms of the Majorana fermion.) The number of Dirac fermion flavors will be denoted by N, which corresponds to $N_f = 2N$ in term of the original Majorana fermions.

Our starting point is the partition function $Z_{[\mathsf{a}]}(g) = \mathrm{Tr}_{a_x,a_y} e^{2\pi i(a_\tau - 1/2)F} e^{-2\pi R_0 H'}$ with H' given by Eq. (3.21). Let us now include the effects of parity symmetry by including twisted boundary conditions by parity. First, note that the modular parameters β and γ are odd under parity. Hence they will be set to zero henceforth, $\beta = \gamma = 0$, to consider the parity twisted partition function. While $\mathrm{SL}(3,\mathbb{Z})$ acts on the metric $g = g(R_i, \alpha, \beta, \gamma)$, there is an $\mathrm{SL}(2,\mathbb{Z})$ subgroup generated by U_1' and U_2, acting on the "reduced" set of the modular parameters, $g_{\mathrm{P}} = g_{\mathrm{P}}(R_i, \alpha) \equiv g(R_i, \alpha, \beta = \gamma = 0)$. [4] With the reduced set of modular parameters by parity, the total partition function, which is generated by projection by parity and the fermion number parity, is given by

$$\begin{aligned} \mathcal{Z}^{tot}(g_{\mathrm{P}}) &= \sum_{\mathsf{G} \in SG^3} \epsilon_{\mathsf{G}} \left[Z_{\mathsf{G}}(g_{\mathrm{P}}) \right]^N, \\ Z_{\mathsf{G}}(g_{\mathrm{P}}) &= \mathrm{Tr}_{G_x, G_y} \left[G_\tau (-1)^F e^{-2\pi R_0 H'} \right], \end{aligned} \tag{3.61}$$

where $SG = \{1, \mathscr{G}_f, \mathscr{P}, \mathscr{P}\mathscr{G}_f\}$ is the symmetry group of the surface fermion theory, and ϵ_{G} are weights assigned to different sectors with partition functions twisted by $\mathsf{G} = (G_\tau, G_x, G_y)$, where for each direction, the boundary condition is twisted by $G_\mu = 1, \mathscr{G}_f, \mathscr{P}, \mathscr{P}\mathscr{G}_f$.

Not all sectors of the total partition function are mixed by $\mathrm{SL}(2,\mathbb{Z})$. We can then divide different sectors into groups, and study the action of $\mathrm{SL}(2,\mathbb{Z})$ on each group separately. In the following, we will focus on the sectors generated by twisting y-boundary condition by $1, \mathscr{G}_f$, and by twisting τ- and x- boundary conditions by $1, \mathscr{G}_f, \mathscr{P}, \mathscr{G}_f\mathscr{P}$. For a given y-boundary condition, there are $4^2 = 16$ sectors in total, and the corresponding partition function is

$$\mathcal{Z}^{tot}_{[a_y]}(g_{\mathrm{P}}) = \sum_{\mathsf{G} \in SG^2} \epsilon_{\mathsf{G}, a_y} \left[Z_{(\mathsf{G}, \mathscr{G}_f^{2a_y})}(g_{\mathrm{P}}) \right]^N, \tag{3.62}$$

where $\mathsf{G} = (G_\tau, G_x)$, and $a_y = 0, 1/2$ represents the y-boundary condition. We will consider the cases of $a_y = 0$ and $a_y = 1/2$ separately, as they are not mixed by $\mathrm{SL}(2,\mathbb{Z})$. The remaining sectors can be generated by twisting y-boundary condition by $\mathscr{P}$ and $\mathscr{P}\mathscr{G}_f$. Twisting by these group elements gives rise to what can be interpreted as "open sectors" (partition functions on orbifolds) as noted by Horava [113]. In this paper, however, we will focus on the 32 "closed" sectors generated by twisting with $G_y = 1, \mathscr{G}_f$. The resulting closed

[4] For Dirac fermions, parity also restricts the possible values of the background flux to be $\mathsf{a}_{\mathrm{P}} \equiv (a_\tau, a_x, a_y = 0, 1/2)$. However, since our theory here is considered as a double theory for Majorana fermions, a_μ takes only 0 or 1/2.

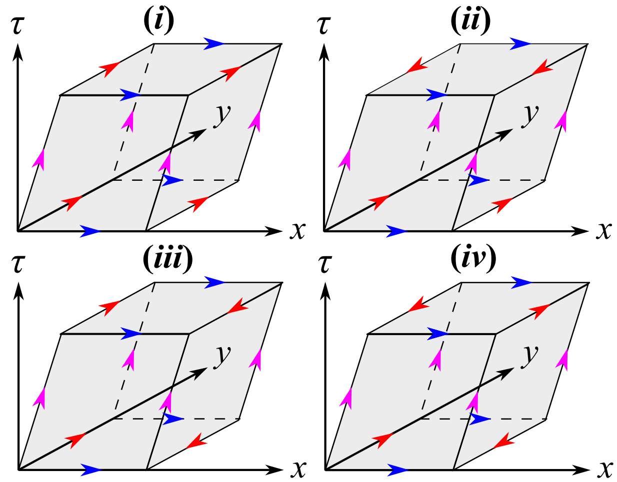

Figure 3.1: The three-torus and its (unorientable) descendants generated by the orientifold projection. While the y-boundary condition is twisted by $G_y = 1$ or $\mathscr{G}_f$, the τ- and x- boundary conditions are twisted by $(G_\tau, G_x) = (\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x})$, $(\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x})$, $(\mathscr{G}_f^{2a_\tau}, \mathscr{P}\mathscr{G}_f^{2a_x})$, $(\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{P}\mathscr{G}_f^{2a_x})$, as shown in figures (i)–(iv), respectively. (Un)twisted boundary conditions are represented by arrows with the same color.

orientable/unorientable three-manifolds, where the (twisted) partition functions are evaluated, are shown in Fig. 3.1.

In the following, we present the analysis of the twisted partition functions for the case of $a_y = 0$. The detail of the calculations is left to Appendix B.4. The analysis for the case of $a_y = 1/2$ is similar and in fact simpler. In short, for $a_y = 1/2$, the total partition function (3.62) can be made modular invariant for any number of Dirac fermion flavors, N. See Appendix B.6.

On the other hand, the total partition function for $a_y = 0$ can or cannot be made modular invariant, depending on N. For $a_y = 0$, there are 16 sectors in total, generated by twisting by $\mathscr{P}$ and $\mathscr{G}_f^{2a_\mu}$ in the τ or/and x directions. We divide these 16 sectors into four sets (i–iv) by $(G_\tau, G_x) = (\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x})$, $(\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x})$, $(\mathscr{G}_f^{2a_\tau}, \mathscr{P}\mathscr{G}_f^{2a_x})$, $(\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{P}\mathscr{G}_f^{2a_x})$, respectively. (There are four sectors in each set.) The symmetry-twisted partition functions $Z_{(G_\tau, G_x, \mathscr{G}_f^{2a_y})}$ for each set are then given by (see Appendix B.4)

$$
\begin{aligned}
\chi^{i}_{[a_\tau,a_x]}(g_{\mathrm{P}}) &= A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x,a_\tau]}(\tau_{2d})\Theta^{i}_{[a_x,a_\tau]}, \\
\chi^{ii}_{[a_\tau,a_x]}(g_{\mathrm{P}}) &= A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x,a_\tau-\frac{1}{2}]}(\tau_{2d})\Theta^{ii}_{[a_x,2a_\tau]}, \\
\chi^{iii}_{[a_\tau,a_x]}(g_{\mathrm{P}}) &= A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x-\frac{1}{2},a_\tau]}(\tau_{2d})\Theta^{iii}_{[2a_x,a_\tau]}, \\
\chi^{iv}_{[a_\tau,a_x]}(g_{\mathrm{P}}) &= A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x-\frac{1}{2},a_\tau-\frac{1}{2}]}(\tau_{2d})\Theta^{iv}_{[2a_x,a_\tau-a_x]},
\end{aligned}
\tag{3.63}
$$

where $a_{\tau,x} = 0, 1/2$ and we have introduced the functions $\Theta^{i-iv}_{[a_x,a_\tau]}(\tau_{2d}; r_{12})$ by

$$\begin{aligned}
\Theta^{i}_{[a_x,a_\tau]} &= \prod_{s_y \in \mathbb{Z}^+} \left[\Theta_{[a_x,a_\tau]}\left(\tau_{2d}; r_{12} s_y\right)\right]^2, \\
\Theta^{ii}_{[a_x,a_\tau]} &= \prod_{s_y \in \mathbb{Z}^+} \Theta_{[a_x,a_\tau]}(2\tau_{2d}; r_{12} s_y), \\
\Theta^{iii}_{[a_x,a_\tau]} &= \prod_{s_y \in \mathbb{Z}^+} \Theta_{[a_x,a_\tau]}(\tau_{2d}/2; 2r_{12} s_y), \\
\Theta^{iv}_{[a_x,a_\tau]} &= \prod_{s_y \in \mathbb{Z}^+} \Theta_{[a_x,a_\tau]}(\tau_{2d}/2 + 1/2; 2r_{12} s_y).
\end{aligned} \tag{3.64}$$

When evaluating the partition sum (3.62), constant prefactors may show up, but are not displayed in the expressions (3.63). These prefactors correspond to parity eigenvalues of the ground states in different sectors [which might depend on the modular parameters and fluxes but are assumed to be $\mathrm{SL}(2,\mathbb{Z})$ invariant], and can be absorbed to the (redefined) weights $\epsilon_{\mathsf{G},a_y}$ in Eq. (3.62).

We now ask, for a specific choice of N, by summing these partition functions with some set of weights, if we can construct a modular invariant. The transformation properties of the twisted partition functions $\chi^{i-iv}_{[a_\tau,a_x]}$ under $\mathrm{SL}(2,\mathbb{Z})$ (generated by U_1' and U_2) can be deduced from the properties of $A^{R,L}$ and Θ shown in Sec. 3.1; see Appendix B.5. It can be shown that if and only if $N = 4n$ ($n = 1, 2, 3, \ldots$), i.e., $N_f = 8n$, a modular invariant can be constructed. In addition, while $\mathrm{SL}(2,\mathbb{Z})$ invariance can be achieved for $N = 4n$, there is a distinction between $n = 2k-1$ and $n = 2k$ ($k = 1, 2, 3, \ldots$), i.e., $N = 8k-4$ ($N_f = 16k-8$) and $N = 8k$ ($N_f = 16k$). To be explicit, the twisted partition functions in set (i) are closed under $\mathrm{SL}(2,\mathbb{Z})$ and a modular invariant can be constructed for any N. For the twisted partition functions in set ($ii-iv$), we consider a weighted sum $\sum_{A=ii,iii,iv}\sum_{i=1}^{4}\epsilon^A_i(\chi^A_i)^N$, where $\chi^A_1 = \chi^A_{[0,0]}$, $\chi^A_2 = \chi^A_{[\frac{1}{2},0]}$, $\chi^A_3 = \chi^A_{[0,\frac{1}{2}]}$, $\chi^A_4 = \chi^A_{[\frac{1}{2},\frac{1}{2}]}$. When $N = 8k-4$, the $\mathrm{SL}(2,\mathbb{Z})$ invariance is achieved when

$$(\epsilon^{ii}_1, \epsilon^{ii}_2, \epsilon^{ii}_3, \epsilon^{ii}_4, \epsilon^{iii}_1, \epsilon^{iii}_2, \epsilon^{iii}_3, \epsilon^{iii}_4, \epsilon^{iv}_1, \epsilon^{iv}_2, \epsilon^{iv}_3, \epsilon^{iv}_4) = (a_1, a_2, a_3, a_3, a_1, a_3, a_2, a_3, -a_1, -a_3, -a_3, -a_2), \tag{3.65}$$

where $a_{i=1,2,3}$ are arbitrary phases (signs). Thus, when $N = 8k-4$, the trivial choice, $\epsilon^A_i = 1$ for all (A, i), is not allowed. When $N = 8k$, on the other hand, the $\mathrm{SL}(2,\mathbb{Z})$ invariance is achieved when

$$(\epsilon^{ii}_1, \epsilon^{ii}_2, \epsilon^{ii}_3, \epsilon^{ii}_4, \epsilon^{iii}_1, \epsilon^{iii}_2, \epsilon^{iii}_3, \epsilon^{iii}_4, \epsilon^{iv}_1, \epsilon^{iv}_2, \epsilon^{iv}_3, \epsilon^{iv}_4) = (a_1, a_2, a_3, a_3, a_1, a_3, a_2, a_3, a_1, a_3, a_3, a_2). \tag{3.66}$$

Dimensional reduction to the edge theory of the (2+1)d fermionic SPT phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The $\mathrm{SL}(2,\mathbb{Z})$ invariance for $N = 4n$ ($N_f = 8n$) may be understood by taking the limit

$R_2 \to 0$ ($r_{12} \to \infty$). In this limit, all massive theta functions become 1 and the total partition function constructed here reduces to the form of the (1+1)d partition function projected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (fermion number parity conservation for each chirality), which is the edge theory of the (2+1)d SPT phase with spin parity conservation [28]. In the latter case, the $\mathrm{SL}(2, \mathbb{Z})$ invariance of the $N_f = 8n$ symmetry-projected partition function indicates that $8n$ helical Majorana edge modes [in (1+1) dimensions] can be gapped without breaking the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

3.5 Discussion

We have studied global anomalies on surface theories of (3+1)d topological insulators and superconductors. For CP symmetric TIs, which are related to, by CPT-theorem, time-reversal symmetric TIs, there is a global U(1) gauge anomaly if the number of the surface Dirac fermion is odd, characterizing the $\mathbb{Z}_2$ classification of the bulk phase. For reflection symmetric TSCs, which are related to, by CPT-theorem, class DIII TSCs, a global gravitational anomaly is present in the surface theory when $N_f \neq 0 \mod 8$. The corresponding bulk state is topologically distinct from trivial states of matter even in the presence of interactions, as far as the bulk gap is not destroyed by the interactions. On the other hand, the weights ϵ_{G}, determining the relative weights among partition functions in different sectors, have 16-periodicity as a function of N_f. Our analysis thus presents an alternative approach to the collapse of the non-interacting classification.

For the cases where we do not find any inconsistency (anomaly), i.e., the case of TSCs with $N_f = 8$, the situation may be more subtle. First of all, the theory may suffer from other forms of inconsistency, which have not been studied here, and hence particular calculations presented in this work does not immediately conclude that the corresponding (3+1)d bulk theories are topologically trivial. Recall that we have not included the partition functions twisted in the y direction by $\mathscr{P}$ and $\mathscr{P}\mathscr{G}_f$ [see comments below Eq. (3.62)].

Moreover, we studied the problem of global anomalies by considering surface theories on T^3 (and its descendants generated by the orientifold projection). Even when the theory is shown to be consistent on T^3, it may be anomalous once put on a different three-manifold. The situation is better understood for 2d conformal field theories (CFTs), where once the consistency of the theories at genus one (torus) is established, they can be consistently defined on any (oriented) Riemann surfaces. For 3d CFTs, there is no such known fact. For this reason, our quest for anomalies in the surface theories may not be complete. Nevertheless, our study on anomalies of 3d massless fermions has shown some interesting and nontrivial results [5].

[5] The questions addressed here, after the completion of this work, was answered in a recent paper by E. Witten [56]. The $N_f = 8$ surface theory of a TSC is actually anomaly-free – in the traditional sense – on any 3-manifolds, either orientable or unorientable. However, such surface state indeed suffers from some other inconsistencies. When one considers the problem of anomalies in a more subtle way (than the situation considered in this paper), the anomaly is of order 16 rather than 8. See the discussion in Ref. [56].

Finally, it is interesting whether our approach can be related to the gapped surface states of (3+1)d SPT phases that develop symmetry-respecting topological orders. Such connection is recently investigated in Ref. [44] in the case of the SU(2) global anomaly [123]. Extending such connection to a generic set of interacting SPT phases is left for future studies.

Chapter 4

Revisiting discrete gauge anomalies in 3+1 dimensions: From the perspective of symmetry-protected topological phases in 4+1 dimensions

4.1 Introduction

Previously, it is argued that the anomaly constraints on $\mathbb{Z}_n$ charges of a set of massless chiral (Weyl) fermions in 3+1 dimensions can be deduced by embedding the $\mathbb{Z}_n$ gauge symmetry in an U(1) gauge group; these constraints are derived basing on the anomaly cancellation conditions of the U(1) symmetry, which involve the perturbative gauge and mixed gauge-gravitational anomalies, together with the constraints on the charges of the fermions that acquire mass through spontaneous breaking of U(1) [60]. Specifically, let $\{\{q_i\}, \{Q_j\}\}$ be the U(1) charges of a collection of right-handed Weyl fermions [1]. To guarantee that the theory is anomaly-free, these charges must obey the relations $\sum_i q_i^3 + \sum_j Q_j^3 = 0$ and $\sum_i q_i + \sum_j Q_j = 0$. We then introduce a Higgs field ϕ of charge n to spontaneously break the U(1) symmetry down to a $\mathbb{Z}_n$ symmetry, and also add Yukawa couplings between the ϕ field and the charge-Q_j fermions, so that these fermions gain mass from the expectation value of ϕ (while the charge-q_i fermions are left massless in the low energy phase). A generic Yukawa coupling includes the Dirac-type mass terms, which couple each pair of different Weyl fermions, and the Majorana-type mass terms, which couple each Weyl fermion with itself. As these mass terms are required to be gauge invariant when coupled to single-valued functions of the Higgs field, the charges of the massive fermions must obey $Q_{j'} + Q_{j''} = \text{integer} \times n$ for each pair of fermions with

[1] The contribution of a left-handed Weyl fermion of charge q to the anomaly is equal to a right-handed Weyl fermon of charge $-q$. Without loss of generality, one can just consider fermions with a specific chirality to derive the anomaly constraints.

a Dirac mass and, if n is even, $2Q_l = \text{integer} \times n$ for each fermion with a Majorana mass. Then, writing $q_i = \tilde{q}_i + m_i n$, where $\tilde{q}_i, m_i \in \mathbb{Z}$ and $0 \leq \tilde{q}_i < n$, the U(1) anomaly cancellation conditions plus the charge constraints on the massive states yield

$$\sum_i \tilde{q}_i^3 = pn + r\frac{n^3}{8}, \quad p, r \in \mathbb{Z};\; p \in 3\mathbb{Z} \text{ if } n \in 3\mathbb{Z},$$
$$\sum_i \tilde{q}_i = p'n + r'\frac{n}{2}, \quad p', r' \in \mathbb{Z}. \tag{4.1}$$

Therefore, for a $\mathbb{Z}_n$ gauge symmetry, the $\mathbb{Z}_n$ charges $\{\tilde{q}_i\}$ of a set of Weyl fermions must satisfy the above condition – the so-called *Ibanez-Ross condition*; it is understood to be necessary but not sufficient, as the $\mathbb{Z}_n$ gauge theory (coupled to Weyl fermions) are assumed to be the low energy theory of an embedding U(1) gauge theory [2]. Also, in this derivation, we have implicitly assumed that all the U(1) charges have integer values and massive fermions (after U(1) is broken) of integer charges do not contribute to cancellation of the anomaly of a low energy $\mathbb{Z}_n$ gauge group.

The constraint (4.1) that is linear in the $\mathbb{Z}_n$ charges can also be argued by considering the violation of the low energy $\mathbb{Z}_n$ symmetry in the presence of a gravitational instanton which is a spin manifold [59, 124], [3] without referring to information of any high energy theories in which the massless fermions are embedded. On the other hand, the nonlinear (cubic) constraint, as pointed out by Banks and Dine in [59], might be too restrictive and might not be required for consistency of the low energy theory, while it is not solely from the low energy considerations and would depend on assumptions about high energy theories. In particular, changes of the normalization of U(1) charges would affect this constraint. The cubic constraint could be weaker if we are not restricted to integer normalization of charges. For examples, one can always make a set of $\mathbb{Z}_n$ charges satisfying the cubic constraint by embedding it in a theory with $\mathbb{Z}_{n^2}$ symmetry, that is, fractional charges with unit $1/n$. However, one can not do so for an anomalous $\mathbb{Z}_n$ symmetry that does not satisfy the linear constraint by adding any massive sector to the low energy theory (when embedded to an U(1) gauge theory), regardless of the normalization of the charges.

Thus, so far it is believed that the linear constraint in (4.1) is more fundamental than the nonlinear one; the former is required for a $\mathbb{Z}_n$ gauge symmetry in a theory of massless fermions, while the latter is not necessary [59]. Failure of the nonlinear constraint implies only the existences of some fractionally charged massive states and an enlarged symmetry group at high energy. (Another point of view is that these fractionally charged states are indeed anomalous and contribute to cancellation of the anomaly of the low

[2] Conversely, for a given set of $\mathbb{Z}_n$ charges $\{\tilde{q}_i\}$ that satisfies the Ibanez-Ross condition (for some integers p, r, p', r'), we are not sure if there always exists an high energy U(1) gauge theory in which such a $\mathbb{Z}_n$ gauge theory can be embedded.

[3] One can also constrain the $\mathbb{Z}_n$ symmetry by introducing gauge instantons of a continuous gauge symmetry to the theory, as argued in [125, 59]. Here we consider the situation that only gravitational instantons are present.

energy states [58]; however, it was not known at that time whether one can present the nonlinear constraint in a way such that it throws much light on the nature of these states.)

In this paper, we revisit the problem of gauging a discrete internal symmetry – $\mathbb{Z}_n$ symmetry in particular – in theories of massless chiral fermions in 3+1 dimensions. Our approach, when focusing only on the low energy degrees of freedom (Weyl fermions) themselves, is based on geometrical considerations. The anomaly constraints on the discrete gauge symmetry are derived by looking at the consistency of formulating fermion theories on any four-dimensional spin manifold endowed with a background gauge field associated to this symmetry. We found, for a $\mathbb{Z}_n$ symmetry, there are both linear and nonlinear (cubic) constraints, so our result agrees with that in [60]. However, we would like to clarify some points (regarding the concern in [59]):

• Since the conditions we derive are independent of assumptions about any high energy embedding theories, they must be the (anomaly) constraints on discrete symmetries in low energy theories when coupled to gravity (geometry). (The consideration of the gravity effects – even if we do not talk about quantum gravity – is natural and reasonable as the spacetime is continuous. However, we do not know whether the same constraints from such consideration are also required for symmetries realized on a lattice system.)

• If one considers the violation of a $\mathbb{Z}_n$ *global* symmetry in the low energy theory in terms of gravitational instantons that are spin manifolds, one would obtain only the linear constraint. Here we consider the obstruction of a $\mathbb{Z}_n$ *gauge* symmetry – as our theory is formulated on spin manifolds with $\mathbb{Z}_n$ gauge bundles – and thus the resulting nonlinear constraint, in additional to the linear one, is also important. Only a $\mathbb{Z}_n$ symmetry that satisfies both of these two kinds of constraints is free from the so-called 't Hooft anomaly that would obstruct gauging it (in a way consistent with gravity). Therefore, as we will see later, the $\mathbb{Z}_n$ gauge anomaly would depend on the normalization of the discrete charges carried by fermions.

Actually, we give a classification of fermion theories with anomalous $\mathbb{Z}_n$ symmetries in 3+1 dimensions, which is represented by an Abelian group in terms of $\mathbb{Z}_n$ charges. Thus the condition for an anomaly-free $\mathbb{Z}_n$ symmetry is naturally obtained by identifying the identity element of this group. Due to the bulk-boundary correspondence [some refs.], the same group also classifies fermionic symmetry-protected topological (SPT) phases with $\mathbb{Z}_n$ symmetry in one dimension higher.

Knowing the anomaly of a given theory of massless fermions, it is possible to introduce extra degrees of freedom (such as matter or gauge fields) and/or interactions to this theory, without changing the anomaly under the same symmetry, so that the whole system becomes gapped. However, the symmetry must be realized projectively, that is, the (total) symmetry group would be enlarged and, in some cases, there are "fractional" charges present in the new theory. Therefore, our analysis of discrete gauge anomalies by geometrical considerations also provides a fundamental understanding of gapped states of fermions with anomalous discrete symmetries.

4.2 Anomalies of discrete symmetries in theories of Weyl fermions in four dimensions

4.2.1 Fermion path integral on 4d spacetime manifolds with discrete background gauge fields

Let $\Psi = (\psi_1, \psi_2, \dots, \psi_N)$ be a set of spin 1/2 Weyl fermions with the same chirality, say, positive chirality, transforming in a representation R of an internal symmetry G in four spacetime dimensions. Here we focus on the case that G is a finite group. As mentioned in the introduction, we would like to study the problem of gauging G, by coupling Ψ to a background gauge field associated to the representation R, on an arbitrary four-dimensional spacetime manifold.

Let us make the above statement more precise. We formulate the fermion theory on a generic compact Riemannian four-manifold (M, g) endowed with a spin structure and a G structure. Here we work in Euclidean signature. We denote such a space by (M, g, s, f), where s is a spin structure parametrized by elements of $H^1(M, \mathbb{Z}_2)$ and f is a classifying map (defined up to homotopy) from M to the classifying space BG that gives a G structure. The Weyl fermions with positive chirality are spinors in a section of the product bundle $S^+(M) \otimes V_R$, where $S^+(M)$ is the positive spinor bundle over M and V_R is an associated vector bundle of the underlying G-bundle over M in the representation R. Note that, as G is a finite group, the transition functions of V_R are locally constant (as V_R is flat) and thus the chiral Dirac operator $\mathcal{D}_R^+$, which maps sections of $S^+(M) \otimes V_R$ to sections of $S^-(M) \otimes V_R$, is locally isomorphic to the form

$$\mathbb{1}_{n\times n} \otimes \left[i\gamma^\mu (\partial_\mu + \omega_\mu) \frac{1+\gamma^5}{2} \right], \tag{4.2}$$

where ω_μ is the spin connection and γ^μ and γ^5 are respectively the curved-space gamma matrices and the chirality matrix in four dimensions.

Now we wonder whether the partition function of the system, evaluated as $\det(\mathcal{D}_R^+)$ on $S^+(M) \otimes V_R$ by some suitable regularization, is well-defined or not, in the meaning of respecting diffeomorphism invariance.

First, since the total Lagrangian density is locally isomorphic to $n = \dim(R)$ copies of the Lagrangian density for a single Weyl fermion without gauge fields, there is no perturbative gravitational anomaly in the theory, while such an anomaly occurs only in $4k+2$ dimensions [126]. That is, the partition function is invariant under infinitesimal diffeomorphisms.

However, the theory may have global anomalies, which in general depend on the topology of twisted spinor bundle $S^+(M) \otimes V_R$ and are typically more difficult (comparing with the perturbative anomalies) to analyze. A traditional definition of global anomalies is given by the non-invariance of the partition function under large diffeomorphisms (or ones combined with gauge transformations if continuous gauge fields are present). These anomalies are represented by U(1) phases that can be evaluated by the (exponentiated) eta-invariants of the five-dimensional Dirac operator on all possible twisted spinor bundles (in the representation R of G) over the mapping tori obtained by gluing together the ends of $M \times [0,1]$ via large diffeomorphisms that preserve both the spin structure and the G structure on M [123, 115].

Yet this is still not the whole story for the problem of anomalies. If there exists any five-dimensional manifold with boundary M such that all the metric, the spin structure, and the gauge field (the G-bundle) on M can extend over it, the theory on M, as a boundary theory of some theory defined on the five-manifold, should not depend on the way it extends in one dimension higher. To be more specific, let X be a five-manifold with boundary $\partial X = M$. Then the Dai-Freed theorem [127] gives a physically sensible definition of the partition function of the whole system [56, 57, 128]

$$Z_\Psi = |\det \mathcal{D}_R^+(M)| \exp(-2\pi i \eta_{\mathrm{Spin},R}(X)). \tag{4.3}$$

Here $\mathcal{D}_R^+(M)$ is the chiral Dirac operator on $S^+(M) \otimes V_R$ described previously and $\eta_{\mathrm{Spin},R}(X)$ is the Atiyah-Patodi-Singer (APS) eta-invariant of the Dirac operator $\mathcal{D}_R(X)$ on a twisted spinor bundle over X that equals $S^+(M) \otimes V_R$ when restricted to M; it is defined as an analytic measure of the spectral asymmetry of $\mathcal{D}_R(X)$:

$$\eta_{\mathrm{Spin},R}(X) = \frac{1}{2} \lim_{s \to 0} \left(\sum_{\lambda \neq 0} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s} + \dim \mathrm{Ker}\,(\mathcal{D}_R(X)) \right), \tag{4.4}$$

where λ are nonzero eigenvalues of $\mathcal{D}_R(X)$ and a regularization of the infinite sum at $s = 0$ is taken. Then, we would like to ask if the formula (4.3) depends on the twisted spinor bundle (over a five-manifold) on which $\eta_{\mathrm{Spin},R}$ is evaluated. If so, the theory of massless fermions Ψ on (M, g, s, f), with the partition function defined via the formula (4.3), is anomalous, in the meaning of being a purely four-dimensional theory; that

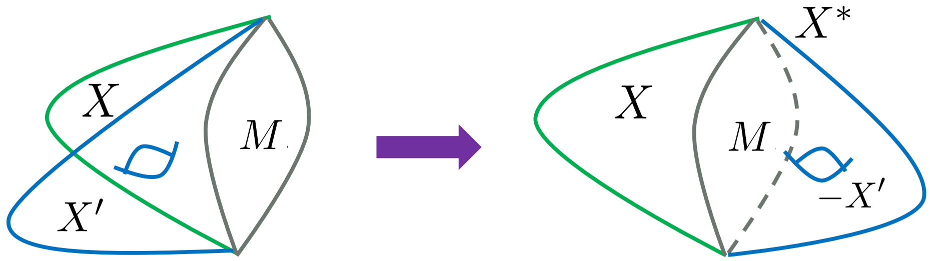

Figure 4.1: Two manifolds X and X' with the same boundary M are glued together along M to make a closed manifold X^*.

is, Z_Ψ includes the contribution from the (bulk) partition function in five dimensions. This is a refined definition of the global anomalies given in [56, 57].

The condition for whether a theory is free from such kind of anomalies can be determined in the following way. Suppose there exist two five-manifolds X and X' with the same boundary M such that the metric and all the structures on M extend over each of them. The two twisted spinor bundles over X and X' restrict to the same twisted spinor bundle $S^+(M) \otimes V_R$ over M. By reversing the orientation of X' and by taking appropriate spin and G structures associated with this reversal, one can then glue X and X' (and all their structures) together along M to make a closed manifold X^*, as shown in FIG. 4.1. Since the eta-invariant respects a gluing law as the usual gluing relation for any local effective action on manifolds (and bundles) [57, 127], one has

$$\begin{aligned}\frac{Z_\Psi}{Z'_\Psi} &= \frac{\exp(-2\pi i \eta_{\mathrm{Spin},R}(X))}{\exp(-2\pi i \eta_{\mathrm{Spin},R}(X'))} \\ &= \exp(-2\pi i \eta_{\mathrm{Spin},R}(X)) \exp(+2\pi i \eta_{\mathrm{Spin},R}(-X')) \\ &= \exp(-2\pi i \eta_{\mathrm{Spin},R}(X^*)). \end{aligned} \tag{4.5}$$

Now it is obvious that Z_Ψ given by the formula (4.3) does not depend on the choice of X and the structures on it if and only if $\exp(-2\pi i \eta_{\mathrm{Spin},R}(X^*))$ equals 1 for any closed five-manifolds X^* endowed with any possible spin and G structures.

As G is finite, $\exp(-2\pi i \eta_{\mathrm{Spin},R}(X^*))$ or $\eta_{\mathrm{Spin},R}(X^*) \mod \mathbb{Z}$ on a closed five-manifold with spin and G structures is a bordism invariant. That is, if X^* bounds a six-dimensional spin manifold Z such that all the structures on X^* extend over Z, the APS index theorem [129, 130] tells us that $\eta_{\mathrm{Spin},R}(X^*)$ equals the index of the Dirac operator on the twisted spinor bundle over Z (with the APS boundary condition) and thus $\eta_{\mathrm{Spin},R}(X^*)$ is an integer. Note that there is no contribution from the local invariant in the bulk of Z to this index, because the Dirac genus of Z, $\hat{A}(Z)$, vanishes in six dimensions. (This also means there is no perturbative gravitational anomalies in the four-dimensional fermion theory, as mentioned before.)

The Dai-Freed theorem gives a natural way to "classify" the anomaly of the four-dimensional massless fermions Ψ in an arbitrary representations R of G, through the eta-invariant map

$$\eta_{\mathrm{Spin},R} : \Omega_5^{\mathrm{Spin}}(BG) \to \mathbb{R}/\mathbb{Z} \quad \text{by} \quad [(X^*, g, s, f)] \mapsto \eta_{\mathrm{Spin},R}(X^*) \mod \mathbb{Z}, \tag{4.6}$$

where $\Omega_5^{\mathrm{Spin}}(BG)$ is the (equivariant) spin bordism group of closed five-manifolds with spin and G structures and we denote elements of $\Omega_5^{\mathrm{Spin}}(BG)$ by the bordism classes of topological spaces $[(X^*, g, s, f)]$ with metrics g, spin structures s, and G structures f. Actually, $\eta_{\mathrm{Spin},R} \mod \mathbb{Z}$ or its exponential $\exp(-2\pi i \eta_{\mathrm{Spin},R})$ is also identified as an element of the fermionic symmetry-protected topological (SPT) phases with symmetry G in five dimensions (or 4+1 dimensions in Lorentz signature). The U(1)-valued topological (bordism) invariant $\exp(-2\pi i \eta_{\mathrm{Spin},R}(X^*))$ is the partition function of an invertible topological quantum field theory (TQFT), which describes a fermionic SPT phase at low energy, on a closed five-dimensional spin manifold X^* endowed with a G structure.

It has been proposed that fermionic SPT phases with a generic symmetry group G in d dimensions can be classified by elements of the group

$$\mathrm{Hom}(\Omega_{d,\mathrm{tors}}^{\mathrm{Spin}}(BG), \mathrm{U}(1)), \tag{4.7}$$

where $\Omega_{d,\mathrm{tors}}^{\mathrm{Spin}}(BG)$ is the torsion subgroup of $\Omega_d^{\mathrm{Spin}}(BG)$, the spin bordism group of closed d-dimensional manifolds with spin and G structures [41]. For $d = 5$ and G being a finite group, $\Omega_{5,\mathrm{tors}}^{\mathrm{Spin}}(BG) = \Omega_5^{\mathrm{Spin}}(BG)$, and the exponential eta-invariant maps $\exp(-2\pi i \eta_{\mathrm{Spin},R})$ for all representations of G generate a subgroup of the spin cobordism group $\Omega^5_{\mathrm{Spin}}(BG) := \mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(BG), \mathrm{U}(1))$, which we denote as

$$\Gamma^5_{\mathrm{Spin}}(BG) \leq \Omega^5_{\mathrm{Spin}}(BG). \tag{4.8}$$

As discussed above, elements of $\Gamma^5_{\mathrm{Spin}}(BG)$ correspond to SPT phases of free fermions in five dimensions, and thus, through the bulk-boundary correspondence, classify the anomalies of theories of massless fermions with symmetry G in four dimensions. It is clear an anomaly-free representation of G corresponds to the identity element of $\Gamma^5_{\mathrm{Spin}}(BG)$.

In general, there might exist manifolds with spin and G structures that, as nontrivial elements of $\Omega_5^{\mathrm{Spin}}(BG)$, can not be detected by $\exp(-2\pi i \eta_{\mathrm{Spin},R})$ for any representations of G; that is, $\exp(-2\pi i \eta_{\mathrm{Spin},R})$ equals 1 when evaluated on these manifolds. In this case, $\Gamma^5_{\mathrm{Spin}}(BG)$ is a proper subgroup of $\mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(BG), \mathrm{U}(1))$;

elements of the latter but not of the former correspond to SPT phases that can not be described by free fermions. Nevertheless, for the case $G = \mathbb{Z}_n$ discussed in this paper, we have (as shown later)

$$\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n) = \Omega^5_{\text{Spin}}(B\mathbb{Z}_n). \tag{4.9}$$

4.2.2 Theory of neutral fermions with $\mathbb{Z}_n$ symmetry

In this section we consider neutral Weyl fermions and $G = \mathbb{Z}_n$. We compute the group $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n)$ and the anomaly α_R (defined later) of Ψ in an arbitrary representation R of $\mathbb{Z}_n$. We denote the eta-invariant $\eta_{\text{Spin},R}(X)$ on a closed five-manifold as $\eta(X, R)$, where we omit the "Spin" label as we are focusing on spin manifolds and also put "R" into the parentheses to avoid messy indices when doing ring operations on representations of $\mathbb{Z}_n$.

Let $\mathbb{Z}_n = \{\lambda \in \mathbb{C} : \lambda^n = 1\}$ be the cyclic group of order n. Let $\rho_s(\lambda) = \lambda^s$ be a one-dimensional representation of $\mathbb{Z}_n$, where s is an integer defined modulo n. Any (reducible) representation R of $\mathbb{Z}_n$ is an element of the unitary group representation ring of $\mathbb{Z}_n$:

$$RU(\mathbb{Z}_n) = \oplus_s \rho_s \cdot \mathbb{Z}. \tag{4.10}$$

We also identify $\mathbb{Z}_n = \mathbb{Z}/\{n \cdot \mathbb{Z}\}$ by sending s to $e^{2\pi i s/n}$. This gives $\mathbb{Z}_n$ the structure of a ring.

To compute $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n)$, we need to consider all bordism classes – as $\exp(-2\pi i\eta(X, R))$ or $\eta(X, R)$ mod $\mathbb{Z}$ are bordism invariants – of five-dimensional spin manifolds with $\mathbb{Z}_n$ structures that can be detected by free fermions, which are described by the Dirac theory. These classes form a subgroup of $\Omega^{\text{Spin}}_5(B\mathbb{Z}_n)$, denoted as

$$\Gamma^{\text{Spin}}_5(B\mathbb{Z}_n) \leq \Omega^{\text{Spin}}_5(B\mathbb{Z}_n), \tag{4.11}$$

and are defined through the following equivalence relation: If X_1 and X_2 are two five-dimensional spin manifolds endowed with $\mathbb{Z}_n$ structures, $X_1 \sim X_2$ if $\eta(X_1 - X_2, R) = 0 \mod \mathbb{Z}$ for all representations $R \in RU(\mathbb{Z}_n)$, and we denote $[X_1]_\eta \in \Gamma^{\text{Spin}}_5(B\mathbb{Z}_n)$ be the equivalence class of X_1 associated with this equivalence relation. Clearly, the group $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n)$ defined previously is the Pontryagin dual of $\Gamma^{\text{Spin}}_5(B\mathbb{Z}_n)$, that is, $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n) = \text{Hom}(\Gamma^{\text{Spin}}_5(B\mathbb{Z}_n), \text{U}(1))$. In general, it is not easy to determine the group $\Gamma^{\text{Spin}}_d(BG)$

(and $\Omega_d^{\text{Spin}}(BG)$) for a generic group G in arbitrary dimensions. Nevertheless, the result in [131] gives a way to evaluate $\Gamma_d^{\text{Spin}}(B\mathbb{Z}_n)$ in terms of the representation theory of $\mathbb{Z}_n$, and we will follow their construction to compute $\Gamma_5^{\text{Spin}}(B\mathbb{Z}_n)$ in this section. Moreover, as their original result focused only on the case $n = 2^v$, we also generalize it to any integer n.

Following [131], we consider the lens space bundles (over S^2), a class of five-dimensional spin manifolds endowed with nontrivial $\mathbb{Z}_n$ structures, to study the eta-invariants. They are quotients of the unit sphere bundle of the Whitney sum of the tensor square of the complex Hopf line bundle H and a trivial complex line bundle $\mathbb{1}$ over S^2

$$X(n; a_1, a_2) := S(H \otimes H \oplus \mathbb{1})/\tau(a_1, a_2), \tag{4.12}$$

where $\tau(a_1, a_2) := \rho_{a_1} \oplus \rho_{a_2}$ is a representation of $\mathbb{Z}_n$ in U(2) and its action (by multiplication by λ^{a_i} on the i-th summand) on the associated unit sphere bundle is fixed-point free, that is, a_1 and a_2 are both coprime to n. By construction, the lens space bundles inherit natural spin structures and $\mathbb{Z}_n$ structures by the identification $\pi_1(X(n; a_1, a_2)) = \mathbb{Z}_n$. $X(n; a_1, a_2)$ has a unique spin structure if n is odd, while it has two inequivalent spin structures if n is even; we fix the spin structure for even n by taking the positive sign of the square root of the determinant line bundle $\det(\rho_{a_1} \oplus \rho_{a_2})$.

The eta-invariant on the lens space bundles can be computed by the following combinatorial formula [132, 131]

$$\eta(X(n; a_1, a_2), R) = \frac{1}{n} \sum_{\lambda \in \mathbb{Z}_n, \lambda \neq 1} \text{Tr}(R(\lambda)) \frac{\lambda^{\frac{1}{2}(a_1+a_2)}(1 + \lambda^{a_1})}{(1 - \lambda^{a_1})^2(1 - \lambda^{a_2})}, \tag{4.13}$$

where $R = \oplus_i \rho_{s_i}$ is an arbitrary representation of $\mathbb{Z}_n$. Using the eta-invariant, one can construct isomorphisms from some additive abelian groups formed by spanning sets (over $\mathbb{Z}$) of lens space bundles, which are subgroups of $\Gamma_5^{\text{Spin}}(B\mathbb{Z}_n)$, to the representation theory of $\mathbb{Z}_n$, from which these Abelian subgroups can be further represented in terms of (direct sums of) cyclic groups. Here we present the main result about these isomorphisms and leave the details of the derivation in the appendix.

We first consider the case that n is a prime power, and then generalize the result to any integer n. In the following discussion, when we write $n = p^v$, we implicitly assume p is a prime number. Let

$$S(n) := \begin{cases} \mathrm{span}_{\mathbb{Z}}\{[X(n;1,1)]_\eta\}, & \text{if } n = 2,3, \\ \mathrm{span}_{\mathbb{Z}}\{[X(n;1,1)]_\eta, [X(n;1,3)]_\eta\}, & \text{if } n = 2^v > 2, \\ \mathrm{span}_{\mathbb{Z}}\{[X(n;1,1)]_\eta, [X(n;1,5)]_\eta\}, & \text{if } n = 3^v > 3, \\ \mathrm{span}_{\mathbb{Z}}\{[X(n;1,1)]_\eta, [X(n;1,3)]_\eta\}, & \text{if } n = p^v,\ p > 3. \end{cases} \tag{4.14}$$

By construction, $S(n) \leq \Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ for each $n = p^v$. Then, as shown in Appendix C.1.1, these Abelian groups are isomorphic to direct sums of some cyclic groups:

$$S(n) \cong \begin{cases} 0, & \text{if } n = 2, \\ \mathbb{Z}_n \oplus \mathbb{Z}_{n/4}, & \text{if } n = 2^v > 2, \\ \mathbb{Z}_{3n} \oplus \mathbb{Z}_{n/3}, & \text{if } n = 3^v, \\ \mathbb{Z}_n \oplus \mathbb{Z}_n, & \text{if } n = p^v,\ p > 3. \end{cases} \tag{4.15}$$

One can further identify the abelian groups $S(n)$ with the spin bordism groups $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ for these values of n by using some spectral sequences that give upper bounds for the orders of $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$. For $n = 2^v$, $|\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)|$ are estimated by the Adams spectral sequence [133]:

$$|\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)| \leq n^2/4, \quad n = 2^v,\ v \geq 1. \tag{4.16}$$

For $n = p^v$ of any odd prime p, $|\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)|$ are estimated by the Atiyah-Hirzebruch spectral sequence [130]:

$$\begin{aligned} |\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)| &\leq \prod_{a+b=5} |\tilde{H}_a(B\mathbb{Z}_n, \Omega_b^{\mathrm{Spin}}(pt))| \\ &= |\tilde{H}_0(B\mathbb{Z}_n, 0)| \cdot |\tilde{H}_1(B\mathbb{Z}_n, \mathbb{Z})| \cdot |\tilde{H}_2(B\mathbb{Z}_n, 0)| \cdot |\tilde{H}_3(B\mathbb{Z}_n, \mathbb{Z}_2)| \cdot |\tilde{H}_4(B\mathbb{Z}_n, \mathbb{Z}_2)| \cdot |\tilde{H}_5(B\mathbb{Z}_n, \mathbb{Z})| \\ &= 1 \cdot |\mathbb{Z}_n| \cdot 1 \cdot 1 \cdot 1 \cdot |\mathbb{Z}_n| \\ &= n^2, \quad n = p^v,\ v \geq 1, \end{aligned} \tag{4.17}$$

where $\tilde{H}_k(B\mathbb{Z}_n, M)$ are the (reduced) homology groups of $B\mathbb{Z}_n$ with coefficients in an abelian group M. Observing the order of $S(n)$ for each $n = p^v$ in (4.15), we then conclude, as $S(n)$ is a subgroup of

$\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n) \leq \Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$,

$$S(n) = \Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n) = \Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n), \quad \forall n = p^v. \tag{4.18}$$

Therefore, the (classes of) lens space bundles $[X(n;1,a)]_\eta$ for $a = 1,3,5$, depending on the value of $n = p^v$, are generators (which are not unique) of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ and also $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$. In this case, we can identify the equivalence classes $[\,\cdot\,]_\eta$ with the bordism classes $[\,\cdot\,]$. In principle, one can use the combinatorial formula (4.13) to compute the values of the eta-invariant on these generators to determine the groups that classify five-dimensional fermionic SPT phases with $\mathbb{Z}_n$ symmetries, that is,

$$\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n) = \mathrm{Hom}(\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n), \mathrm{U}(1)) = \mathrm{Hom}(\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n), \mathrm{U}(1)) = \Omega^5_{\mathrm{Spin}}(B\mathbb{Z}_n), \quad \forall n = p^v. \tag{4.19}$$

However, (4.13) is somehow not very useful if we want to look at the dependence of elements of $\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n)$ on representations of $\mathbb{Z}_n$. Instead of using the formula (4.13), one can compute the mod $\mathbb{Z}$ eta-invariant, which is a bordism invariant, in terms of the more familiar A-roof polynomials that appear in the index theorem for Dirac operators. This can be shown by relating the mod $\mathbb{Z}$ eta-invariant on an element of $S(n)$ to the mod $\mathbb{Z}$ eta-invariant on a seven-dimensional lens space, which can be computed using the A-roof polynomials [134]. Again, we leave the derivation in the appendix and only present the result here.

Let

$$\hat{A}_k(t;\vec{x}) := \sum_{a+2b=k} t^a \hat{A}_b(\vec{x})/a!, \tag{4.20}$$

where $\hat{A}_k(\vec{x})$ are the A-roof polynomials. For example,

$$\hat{A}_0(\vec{x}) = 1, \quad \hat{A}_1(\vec{x}) = -\frac{1}{24}\sum_i x_i^2, \quad \hat{A}_2(\vec{x}) = \frac{1}{5760}\left[-4\sum_{i<j} x_i^2 x_j^2 + 7(\sum_i x_i^2)^2\right]. \tag{4.21}$$

Then, as shown in Appendix C.1.2, the mod $\mathbb{Z}$ eta-invariant of (the bordism class of) the lens space bundle $X_n := X(n;1,1)$ in a generic representation $R = \oplus_i \rho_{s_i}$ (recall that ρ_{s_i} is a one-dimensional representation defined by $\rho_{s_i}(\lambda) = \lambda^{s_i},\ \lambda \in \mathbb{Z}_n$) can be expressed as

$$\begin{aligned}\eta([X_n]_\eta, R) &= \frac{1}{n}\sum_i \Big[\hat{A}_4(s_i + 1 + n/2; n,1,1,1,-1) - \hat{A}_4(s_i - 1 + n/2; n,1,1,1,-1)\Big] \mod \mathbb{Z} \\ &= \frac{1}{6n}\sum_i \left(2s_i^3 + 3ns_i^2 + n^2 s_i\right) \mod \mathbb{Z}.\end{aligned} \tag{4.22}$$

One can check that $[X_n]_\eta$ is a generator of a Sylow p-subgroup, that is, a maximal p-subgroup, of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ for each $n = p^v$ by taking a simple representation $R = \rho_1$ and by evaluating the corresponding mod $\mathbb{Z}$ eta-invariant by the above formula:

$$\eta([X_n]_\eta, \rho_1) = \frac{1}{6n}(n+2)(n+1) \quad \text{mod } \mathbb{Z}. \tag{4.23}$$

For $n = 2$, $\eta([X_2]_\eta, \rho_1)$ is an integer, which agrees with the fact $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_2) = 0$. For $n = p^v > 2$, it is obvious that the least integer ℓ for $\ell \cdot \eta([X_n]_\eta, \rho_1)$ to be an integer is $\ell = n$, if n is not divisible by 3, or $\ell = 3n$, if n is divisible by 3. So the order of a subgroup of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ generated by $[X_n]_\eta$ can be determined. Comparing with (4.15) and also recalling that $S(n) = \Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$, we then know $[X_n]_\eta$ is a generator of a Sylow p-subgroup of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$, and moreover, $R = \rho_1$ is also a generator of a Sylow p-subgroup of the dual group $\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n)$, for any $n = p^v$.

On the other hand, we also show (in Appendix C.1.2) that there exists a bordism class $[Y_n]_\eta$ which has the mod $\mathbb{Z}$ eta-invariant

$$\begin{aligned}\eta([Y_n]_\eta, R) &= \frac{1}{n}\sum_i \left[\hat{A}_4(s_i + 2 + n/2; n, 1, 1, 1, -1) - \hat{A}_4(s_i - 2 + n/2; n, 1, 1, 1, -1)\right] \quad \text{mod } \mathbb{Z} \\ &= \frac{1}{3n}\sum_i \left[2s_i^3 + (n^2+6)s_i\right] \quad \text{mod } \mathbb{Z},\end{aligned} \tag{4.24}$$

such that $[X_n]_\eta$ and $[Y_n]_\eta$ form a generating set of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$, for any $n = p^v$.

The expressions (4.22) and (4.24) present the dependence of elements of $\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n)$ on representations of $\mathbb{Z}_n$ in a more transparent way than (4.13). In fact, one can further simplify these expressions through a homomorphism from the abelian group $\{(\eta([X_n]_\eta, R),\ \eta([Y_n]_\eta, R)) \ : \ R \in RU(\mathbb{Z}_n)\}$ [4] to a finite abelian group $A(n)$ that is isomorphic to $S(n)$ by sending the mod $\mathbb{Z}$ eta-invariants associated to a one-dimensional representation ρ_s, that is, $(\eta([X_n]_\eta, \rho_s),\ \eta([Y_n]_\eta, \rho_s))$, to an element

$$\begin{cases} \left(s^3 \mod 1,\ s^3 - s \mod 1\right), & \text{if } n = 2, \\ \left(s^3 \mod n,\ s^3 - s \mod n/2\right), & \text{if } n = 2^v > 2, \\ \left(s^3 \mod 3n,\ s^3 - s \mod n\right), & \text{if } n = 3^v, \\ \left(s^3 \mod n,\ s^3 - s \mod n\right), & \text{if } n = p^v,\ p > 3, \end{cases} \tag{4.25}$$

of $A(n)$ for each $n = p^v$. One can verify that each of these homomorphisms is actually an isomorphism. For

[4] This is an additive abelian group under the direct sum of representations as the addition operation and is finite as we consider the mod $\mathbb{Z}$ eta-invariants.

example, for $n = 2^v > 2$ one has

$$\begin{aligned} s=1: &\quad \left(\frac{n^2+3n+2}{6n} \mod \mathbb{Z},\ \frac{n^2+8}{3n} \mod \mathbb{Z}\right) \to (1 \mod n,\ 0 \mod n/2)\,, \\ s=2: &\quad \left(\frac{n^2+6n+8}{3n} \mod \mathbb{Z},\ \frac{2n^2+28}{3n} \mod \mathbb{Z}\right) \to (8 \mod n,\ 6 \mod n/2)\,. \end{aligned} \tag{4.26}$$

By taking some linear combination of these two equations (which is allowed as it is a group homomorphism) one immediately has

$$(0 \mod \mathbb{Z},\ -2/n \mod \mathbb{Z}) \to (0 \mod n,\ 1 \mod n/2)\,. \tag{4.27}$$

For arbitrary s,

$$\left.\begin{array}{l} \left(x\cdot\frac{n^2+3n+2}{6n} \mod \mathbb{Z},\ x\cdot\frac{n^2+8}{3n} - y\cdot\frac{2}{n} \mod \mathbb{Z}\right) \\ = \left(\frac{sn^2+3s^2n+2s^3}{6n} \mod \mathbb{Z},\ \frac{sn^2+2s^3+6s}{3n} \mod \mathbb{Z}\right) \end{array}\right\} \to (x \mod n,\ y \mod n/2)\,, \tag{4.28}$$

and one solves for a unique solution $x = s^3$ and $y = s^3 - s$ and thus it is a bijective homomorphism, that is, an isomorphism. Cases of $n = p^v$ for other prime powers can also be verified in a similar way.

The result obtained so far can be generalized to any integers n, not just for prime powers. This is essentially based on the the following property of bordism groups [130]

$$\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_{mn}) \cong \Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_m) \oplus \Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n), \quad \text{if } \gcd(m,n) = 1. \tag{4.29}$$

Clearly, the abelain groups $S(n)$ and $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ also satisfy the above property, and correspondingly we have

$$\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_{mn}) \cong \Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_m) \times \Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n), \quad \text{if } \gcd(m,n) = 1, \tag{4.30}$$

for the Pontryagin dual of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$. Now, for any integer written as $n = 2^q \cdot 3^r \cdot k^s$, where $k \geq 5$ is an odd number not divisible by 3 and q, r, s are nonnegative integers, we define the associated integers $a(n)$,

$b(n)$, and $c(n)$ with n respectively

$$a(n) := \begin{cases} k^s, & \text{if } q = 0, 1 \ \& \ r = 0, \\ 3^{r+1} \cdot k^s, & \text{if } q = 0, 1 \ \& \ r \geq 1, \\ 2^q \cdot k^s, & \text{if } q \geq 2 \ \& \ r = 0, \\ 2^q \cdot 3^{r+1} \cdot k^s, & \text{if } q \geq 2 \ \& \ r \geq 1, \end{cases}$$

$$b(n) := \begin{cases} 3^r \cdot k^s, & \text{if } q = 0, \\ 2^{q-1} \cdot 3^r \cdot k^s, & \text{if } q \geq 1, \end{cases}$$

$$c(n) := \begin{cases} k^s, & \text{if } q = 0, 1 \ \& \ r = 0, \\ 3^{r-1} \cdot k^s, & \text{if } q = 0, 1 \ \& \ r \geq 1, \\ 2^{q-2} \cdot k^s, & \text{if } q \geq 2 \ \& \ r = 0, \\ 2^{q-2} \cdot 3^{r-1} \cdot k^s, & \text{if } q \geq 2 \ \& \ r \geq 1. \end{cases} \tag{4.31}$$

Then we have

$$\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n) = \Omega_5^{\mathrm{Spin}}(B\mathbb{Z}_n) \cong \mathbb{Z}_{a(n)} \oplus \mathbb{Z}_{c(n)} \tag{4.32}$$

and

$$\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n) = \Omega^5_{\mathrm{Spin}}(B\mathbb{Z}_n) \cong \mathbb{Z}_{a(n)} \times \mathbb{Z}_{c(n)}, \tag{4.33}$$

which are deduced from (4.15), (4.29), and (4.30). An element of $\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n)$ associated with a generic representation $R = \oplus_i \rho_{s_i}$ of $\mathbb{Z}_n$, which corresponds to a five-dimensional fermionic SPT phase or to the anomaly of a four-dimensional Weyl fermions in the same representation R, can be represented by $\exp(-2\pi i \eta_R)$, where η_R is the eta-invariant map and can be represented by a set of mod $\mathbb{Z}$ eta-invariants

$$(\eta([X_n]_\eta, R),\ \eta([Y_n]_\eta, R)) \tag{4.34}$$

evaluated on generators $[X_n]_\eta$ and $[Y_n]_\eta$ with values given by (4.22) and (4.24) (which apply for any integer n, not just for a prime power p^v). Basing on the discussion around (4.25), one can show, for any integer n, that η_R can also be represented by

$$\alpha_R := \left(\sum_i s_i^3 \mod a(n),\ \sum_i (s_i^3 - s_i) \mod b(n) \right), \tag{4.35}$$

which is an element of the abelian group $\mathbb{Z}_{a(n)} \oplus \mathbb{Z}_{c(n)}$. Then, from this simple expression of α_R, it is clear that a representation $R = \oplus_i \rho_{s_i}$, for a given integer n, is said to be anomaly-free *if and only if*

$$\sum_i s_i^3 = \ell \cdot a(n), \quad \sum_i s_i = m \cdot b(n), \quad \ell, m \in \mathbb{Z}, \tag{4.36}$$

where we have used the fact that, by definition, $a(n)$ can be divided by $b(n)$ for any n.

Connection to the Ibanez-Ross condition

It is not difficult to see that the Ibanez-Ross condition (4.1), which is derived from embedding a $\mathbb{Z}_n$ gauge theory (coupled to Weyl fermions) in a U(1) gauge theory, is consistent with our derivation of the anomaly-free condition (4.36) based on the Dai-Freed theorem and the topological classification of five-dimensional spin manifolds with $\mathbb{Z}_n$ bundles. However, our result gives a complete answer for understanding the 't Hooft anomaly of a $\mathbb{Z}_n$ symmetry, since our derivation only depends on the underlying theories, that is, massless fermions coupled to (background) $\mathbb{Z}_n$ gauge fields, and our condition (4.36) is both necessary and sufficient for an anomaly-free $\mathbb{Z}_n$ symmetry. In contrast, the Ibanez-Ross condition comes from the anomaly constraints of an embedding U(1) gauge theory at high energy, and thus it is only a necessary condition for an anomaly-free $\mathbb{Z}_n$ symmetry at low energy [5]. In particular, for any given anomaly-free representation of $\mathbb{Z}_n$ satisfying (4.36), it is not clear if one can always find an anomaly-free parent U(1) theory whose low energy phase (the massless degrees of freedom) is exactly this $\mathbb{Z}_n$ theory.

Furthermore, in the derivation of the Ibanez-Ross condition, it is implicitly assumed that massive fermions (after U(1) is broken to $\mathbb{Z}_n$) with integer charges do not contribute to cancellation of the $\mathbb{Z}_n$ anomaly in the low energy phase. This assumption is natural but needs to be clarified. From the result in the last section, it can be verified that the $\mathbb{Z}_n$ charges of massive fermions, which satisfy the constraints $Q_{j'} + Q_{j''} = \text{integer} \times n$ for any two particles paired up with a Dirac mass and $2Q_l = \text{integer} \times n$ for any particle with a Majorana mass, indeed do not possess an 't Hooft anomaly, as (by Eq. (4.35))

$$\alpha_{\text{massive}} = \left(\sum_j Q_j^3 \mod a(n),\ \sum_j (Q_j^3 - Q_j) \mod b(n) \right) = (0 \mod a(n),\ 0 \mod b(n)) \,. \tag{4.37}$$

[5] In principle, one needs to consider any possible embedding theories – not just U(1) gauge theories – at high energy to obtain the complete anomaly constraints.

4.3 Anomalous gapped states of fermions in four dimensions

So far, we have been focusing on the problem of gauging a finite global symmetry $G = \mathbb{Z}_n$ in a theory of chiral fermions (either neutral or U(1) charged) coupled to gravity. We also classified the associated anomalous theories, by considering them as boundary states of SPT phases with the same symmetry in one dimension higher, in terms of representations of G. Then, with the same geometrical consideration and knowledge of the anomalies associated with G, one can construct gapped states, starting from states of chiral fermions, that preserve G but still possess the same anomalies as those gapless chiral theories. Note that a theory with a global symmetry that has perturbative anomalies can not be gapped in a symmetry-preserving way. Here we are considering the one which has only global anomalies [6] (if the symmetry is anomalous), so it is possible to realize an anomalous symmetric gapped state. In this section, we present an approach, based on the idea in [135, 57], for constructing gapped boundary states of a nontrivial fermionic SPT phase with symmetry G in five dimensions. Our result thus sheds light on the nature of states of massive fermions with anomalous discrete symmetries in four dimensions.

4.3.1 Some information from $\Gamma^5_{\mathrm{Spin}}(BG)$

Before giving specific models of gapped boundary states, let us see what we can know about the features of these states from the groups $\Gamma^5_{\mathrm{Spin}}(BG)$. From the definitions in Sec. 4.2, these groups give the classification of 4+1-dimensional SPT phases of free fermions with symmetry G. In general, they can be represented in terms of representations of G, as we have shown for $G = \mathbb{Z}_n$. (Though we will mainly focus on the case $G = \mathbb{Z}_n$ in this paper, the argument presented here also applies for a generic symmetry group G.)

For a theory of fermions in any representation of G that corresponds to the identity element of $\Gamma^5_{\mathrm{Spin}}(BG)$, the bulk phase is topologically trivial (what this means is that the TQFT partition function is always 1 when the theory is put on an arbitrary closed and oriented five-manifold endowed with a spin/spin^c structure and a G-bundle), and any associated boundary state is anomaly-free (while one formulates the theory on an open manifold with those structures). In this case, a gapless boundary state represented by chiral fermions can be turned into [7] a gapped, symmetry-preserving, and topologically trivial state, while the bulk phase (in the low-energy limit) is unchanged.

On the other hand, a theory specified by a nontrivial (non-identity) element $\exp(-2\pi i \eta_R)$ (here we omit

[6] For theories of neutral Weyl fermions with a finite global symmetry, there can only be global anomalies. For theories of U(1) charged Weyl fermions, one has to make sure that the perturbative U(1) gauge and mixed gauge-gravitational anomalies are absent.

[7] Here we are allowed to add extra degrees of freedom (such as matter and gauge fields) and/or interactions which are only present on the boundary for constructing a gapped boundary state.

the Spin label) of $\Gamma^5_{\text{Spin}}(BG)$ has a nontrivial bulk phase and, while a boundary is present, an anomalous boundary state, which is in general not unique. The standard boundary state consist of massless chiral fermions. Then, we would like to ask when can such a boundary state be gapped without symmetry breakdown on the boundary. An observation is that if there exists a nontrivial group extension

$$1 \to \mathcal{K} \to \mathcal{H} \to G \to 1, \tag{4.38}$$

where $\mathcal{K}$ is an emergent finite gauge group only coupled to boundary fermions, such that the pullback (associated with the homomorphism $\mathcal{H} \to G$) of the nontrivial element $\exp(-2\pi i \eta_R) \in \Gamma^5_{\text{Spin}}(BG)$ becomes the identity element of $\Gamma^5_{\text{Spin}}(B\mathcal{H})$, then the standard gapless boundary state can be driven to, by the gauge interaction associated to $\mathcal{K}$, a gapped state that respects a global symmetry $\mathcal{H}/\mathcal{K} \cong G$. (A similar discussion for bosonic theories by using cohomology classes is presented in [57].) The gapless and the gapped boundary states have the same anomaly of G, as they are both coupled to the same bulk SPT phase. Note, however, that such an anomalous gapped state is topologically nontrivial, since there is a $\mathcal{K}$ gauge symmetry present in the low energy boundary theory, and the global symmetry G is realized projectively on the boundary.

Let us look at an example. Since we have an explicit expression for $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_n)$, we can take $G = \mathbb{Z}_n$, $\mathcal{K} = \mathbb{Z}_m$, and $\mathcal{H} = \mathbb{Z}_{mn}$ for neutral fermions, where $(m, n) \neq 1$ [8]. A nontrivial extension is specified by the following expressions for generators of these groups:

$$\begin{aligned}
\hat{S} &= \exp(2\pi i \hat{s}/n) \in G = \mathbb{Z}_n, \\
\hat{K} &= \exp(2\pi i \hat{k}/m) \in \mathcal{K} = \mathbb{Z}_m, \\
\hat{H} &= \hat{S} \cdot \hat{K}^{1/n} = \exp(2\pi i (m\hat{s} + \hat{k})/mn) \in \mathcal{H} = \mathbb{Z}_{mn},
\end{aligned} \tag{4.39}$$

where $\hat{s}$ and $\hat{k}$ are the (discrete) charge operators associated with $\mathbb{Z}_n$ and $\mathbb{Z}_m$ symmetries, respectively. Note that the generator of $\mathcal{H}$ satisfies $\hat{H}^n = \hat{S}^n \cdot \hat{K} = \exp(2\pi i(\hat{s} + \hat{k}/m)) \in \mathbb{Z}_m$, so we indeed have $\mathcal{H}/\mathcal{K} \cong G$. Now, we want to know when a nontrivial element $\exp(-2\pi i \eta_R) \in \Gamma^5_{\text{Spin}}(B\mathbb{Z}_n)$, with a generic representation $R = \oplus_i e^{2\pi i s_i/n}$, will be pulled back to the identity element of $\Gamma^5_{\text{Spin}}(B\mathbb{Z}_{mn})$. Instead of considering $\exp(-2\pi i \eta_R)$, we can equivalently take α_R (defined by Eq. (4.35)) for the discussion. Then the

[8] If $(m, n) = 1$, $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$, and the group extension (4.38) is actually trivial.

pullback gives

$$\alpha_R = \left(\sum_i s_i^3 \mod a(n),\ \sum_i (s_i^3 - s_i) \mod b(n) \right)$$
$$\to \left(\sum_I (ms_I + k_I)^3 \mod a(mn),\ \sum_I [(ms_I + k_I)^3 - (ms_I + k_I)] \mod b(mn) \right). \tag{4.40}$$

Here we use a (possibly) different set $\{I\}$ from $\{i\}$ for the representation in the pullback element because a change of the boundary degrees of freedom (with the anomaly unchanged) is possible.

Since $\mathcal{K}$ is a gauge group that only appears on the boundary of a 4+1-dimensional SPT phase with G, there are constraints on these $\mathbb{Z}_m$ charges $\{k_I\}$. For simplicity, let $\{I\}$ be the union of two distinct sets $\{i\}$ and $\{j\}$ and let $s_j = 0$ for all j and $k_i = 0$ for all i. Then the constraints for these gauge charges are $\sum_j k_j^3 = 0 \mod a(m)$ and $\sum_j k_j = 0 \mod b(m)$, as concluded by Eq. (4.36). However, since G is lifted to $\mathcal{H}$ by $\mathcal{K}$ through an element $\hat{K}^{1/n}$, which is also a gauge symmetry (which is broken down to $\hat{K} \in \mathbb{Z}_m$ at low energy), we should also require $\sum_j k_j^3 = 0 \mod a(mn)$ and $\sum_j k_j = 0 \mod b(mn)$. (If the emergent gauge group $\mathcal{K}$ comes from the breakdown of a continuous gauge group, say, U(1), at high energy, the charges k_j must satisfy $\sum_j k_j^3 = 0$ and $\sum_j k_j = 0$ exactly.) Under these constraints, a solution for m and $\{k_I\}$ that makes the pullback element in (4.40) trivial exists only when

$$\sum_i s_i = 0 \mod b(n), \tag{4.41}$$

and one can take, for example, $m^3 = 0 \mod a(mn)$ as a solution. It is noted that, if $\sum_i s_i^3 \neq 0 \mod a(n)$, it is impossible to trivialize α_R when m and n are coprime. This is why we need to consider a nontrivial group extension for trivializing a boundary anomaly.

Here is an interesting observation. When we are given a nontrivial extension of $G = \mathbb{Z}_n$ to $\mathcal{H} = \mathbb{Z}_{mn}$, only a gapless (boundary) theory in a representation of G that has a vanishing "linear $\mathbb{Z}_n$ anomaly" (that is, $\sum_i s_i = 0 \mod b(n)$) can be gapped in a symmetry-preserving way. This conclusion holds for a more general charge assignment $\{s_I, k_I\}$ than that in the above discussion. However, we do not know if it is still true for any nontrivial extension of $G = \mathbb{Z}_n$ to a generic group $\mathcal{H}$.

In the next section, we give a physical model to realize gapped boundary states, and the idea presented here will be more concrete.

4.3.2 A model of anomalous gapped states via weak coupling

Now we present a model of anomalous gapped boundary states in the framework of weak coupling. Our approach is similar to that in [60] for deriving the Ibanez-Ross condition for a $\mathbb{Z}_n$ gauge symmetry, which is briefly reviewed in the introduction, and is a 3+1-dimensional analog of that in [135, 57] for constructing gapped boundary states in 2+1 dimensions.

According to the analysis in the last section, one way to construct a gapped and symmetry-preserving boundary state is to lift the global symmetry G to a group $\mathcal{H}$ by a gauge group $\mathcal{K}$ on the boundary. Note that such a group extension has to be nontrivial. To do this, one can begin with a trivial group extension of G by an enlarged gauge group $\mathcal{G}$ from $\mathcal{K}$

$$1 \to \mathcal{G} \to \mathcal{G} \times G \to G \to 1,$$

and then breaks $\mathcal{G} \times G$ down to $\mathcal{H}$ through a spontaneous symmetry breaking of $\mathcal{G}$ to its subgroup $\mathcal{K}$ at low energy. If the gauge symmetry $\mathcal{G}$ appears only on the four-dimensional boundary, that is, if there is no additional anomaly than the one associated with the global symmetry G when tensoring it with $\mathcal{G}$, then the corresponding low energy theory after symmetry breaking would also possess the same anomaly associated with G.

Here we take $G = \mathbb{Z}_n$ and $\mathcal{G} = \mathrm{U}(1)$ in our model. We first consider the case of (electrically) neutral fermions. As just mentioned, we have to determine in what representation of $\mathcal{G}$, while we are given a set of chiral fermions (on the boundary) in a representation of R of $\mathbb{Z}_n$, there is no extra gauge anomaly and only the $\mathbb{Z}_n$ anomaly (represented by α_R in (4.35)) is present. In general, this is not easy to compute; besides the gauge and mixed gauge-gravitational anomalies for U(1) itself, we also have to take the "mixed anomalies" between U(1) and $\mathbb{Z}_n$ into account. (To be more precise, we have to compute all the anomalies when formulating the theory on a generic spin manifold endowed with a $\mathrm{U}(1) \times \mathbb{Z}_n$-bundle.) Instead of looking at the most general case, we consider a representation

$$\mathbb{1}_{\mathrm{U}(1)} \otimes R \oplus \mathcal{R} \otimes \mathbb{1}_{\mathbb{Z}_n}, \tag{4.42}$$

where $\mathcal{R}$ is a representation of U(1) and $\mathbb{1}_{\mathrm{U}(1)}$ (with the dimension equal to R) and $\mathbb{1}_{\mathbb{Z}_n}$ (with the dimension equal to $\mathcal{R}$) are respectively trivial representations of U(1) and $\mathbb{Z}_n$. To make sure that such a representation has the same $\mathbb{Z}_n$ anomaly as R, we only need to check there is no (perturbative) gauge and mixed gauge-gravitational anomalies for U(1).

Specifically, let $\{\psi_i\}$ and $\{\chi_j\}$ be two sets of Weyl fermions with positive chirality transforming in the representations $\mathbb{1}_{\mathrm{U}(1)} \otimes R$ and $\mathcal{R} \otimes \mathbb{1}_{\mathbb{Z}_n}$ of the group $\mathrm{U}(1) \times \mathbb{Z}_n$, respectively. Here again we let $R = \oplus_i \rho_{s_i}$, where ρ_{s_i} is a one-dimensional representation with a $\mathbb{Z}_n$ charge s_i, and also let $\{k_j \in \mathbb{Z}\}$ be the U(1) charges associated to $\mathcal{R}$, which satisfy the anomalies constraints $\sum_j k_j^3 = 0$ and $\sum_j k_j = 0$. The Lagrangain of the fermions coupled to an emergent U(1) gauge field a that propagates only along the boundary is given by

$$L_0 = \sum_i \overline{\psi}_i i \not{D}_0 P_+ \psi_i + \sum_j \overline{\chi}_j (i\not{D}_0 + k_j \not{a}) P_+ \chi_j, \tag{4.43}$$

where $P_+ = (1+\gamma_5)/2$ and $\not{D}_0$ is the Dirac operator for a fermion coupled to gravity only. In the absence of the U(1) gauge symmetry, the fermions $\{\chi_j\}$ can be fully gapped – as each of them can have a Majorana mass $m_j \overline{\chi}_j \chi_j^c$ – and thus the low energy theory is just the standard boundary state, described by the massless chiral fermions $\{\psi_i\}$, of a 4+1-dimensional fermionic $\mathbb{Z}_n$ SPT phase represented by α_R in (4.35).

In the presence of the U(1) gauge symmetry, the usual mass terms are forbidden, and the part of L_0 that includes the χ_j fermions describes a chiral gauge theory. Nevertheless, one can introduce Higgs fields (charge scalar fields) and Yukawa couplings between them and the fermions to L_0, so that all the fermions receive masses from the expectation values of the Higgs fields. Here we consider a one-Higgs model with a (nonlinear) Yukawa coupling term

$$L_{\mathrm{Yuk}} = \sum_{\mathrm{pairs}\{i,i'\},\ \mathrm{pairs}\{j',j''\},\ l} \left\{ \lambda_{i,i'} \phi^{\alpha_{i,i'}} \overline{\psi}_i \chi_{i'}^c + g_{j',j''} \phi^{\beta_{j',j''}} \overline{\chi}_{j'} \chi_{j''}^c + h_l \phi^{\gamma_l} \overline{\chi}_l \chi_l^c \right\} + \mathrm{h.c.}, \tag{4.44}$$

for some coupling constants $\lambda_{i,i'}$, $g_{j',j''}$, and h_l. Here $\alpha_{i,i'}$, $\beta_{j',j''}$, and γ_l are nonnegative integers and $\chi_j^c := i\gamma^0 \mathcal{C} \chi_j^*$, with $\mathcal{C}$ being the charge conjugation matrix, are the charge conjugate fields of χ_j (which have negative chirality). We have also divided the set of fermions $\{\chi_j\}$ into four distinct sets $\{\chi_{i'}\}$, $\{\chi_{j'}\}$, $\{\chi_{j''}\}$, and $\{\chi_l\}$ (here we assume the number of χ_j is not smaller than the number of ψ_i), such that each $\chi_{i'}$ is paired up with each ψ_i with a Dirac-type mass, each $\chi_{j'}$ is paired up with each $\chi_{j''}$ with a Dirac-type mass, and each χ_l itself is with a Majorana-type mass. Denote the $\mathbb{Z}_n$ and the U(1) charges of ϕ be $\bar{s}$ and $\bar{k}$, respectively. Now, we require L_{Yuk} to be invariant under both the $\mathbb{Z}_n$ global symmetry and the U(1) gauge symmetry, which would constrain the values of the $\mathbb{Z}_n$ and the U(1) charges of all particles:

$$\begin{aligned} \text{Invariant under } \mathbb{Z}_n: &\quad \alpha_{i,i'}\bar{s} - s_i \in n\mathbb{Z}, \quad \beta_{j',j''}\bar{s} \in n\mathbb{Z}, \quad \gamma_l \bar{s} \in n\mathbb{Z}. \\ \text{Invariant under } \mathrm{U}(1): &\quad \alpha_{i,i'}\bar{k} = k_{i'}, \quad \beta_{j',j''}\bar{k} = k_{j'} + k_{j''}, \quad \gamma_l \bar{k} = 2k_l. \end{aligned} \tag{4.45}$$

Substituting the above conditions to the anomalies constraints on k_j, that is, $\sum_j k_j^3 = 0$ and $\sum_j k_j = 0$, we have

$$\sum_i (\bar{k}s_i)^3 = p \cdot \bar{k}n + r \cdot \frac{(\bar{k}n)^3}{8}, \quad p, r \in \mathbb{Z};\ p \in 3\mathbb{Z} \text{ if } n \in 3\mathbb{Z},$$
$$\sum_i s_i = p' \cdot n + r' \cdot \frac{n}{2}, \quad p', r' \in \mathbb{Z}, \tag{4.46}$$

which can also be written, in terms of $a(n)$ and $b(n)$ defined in (4.31), as

$$\sum_i (\bar{k}s_i)^3 = \ell \cdot a(\bar{k}n), \quad \sum_i s_i = m \cdot b(n), \quad \ell, m \in \mathbb{Z}. \tag{4.47}$$

When the field ϕ has a nonzero expectation value $\langle\phi\rangle$, the theory becomes gapped, and the U(1) gauge group is broken down to a finite subgroup $\mathbb{Z}_{\bar{k}}$, as ϕ carries a U(1) charge $\bar{k}$. On the other hand, the original (microscopic) $\mathbb{Z}_n$ global symmetry is also broken, since $\langle\phi\rangle$ is not invariant under a generator $\hat{S} \in \mathbb{Z}_n$ (as $\bar{s}$ is in general not equal to 0 modulo n). For convenience, let us assume $\bar{s} = -1$. Nevertheless, such a gapped phase respects another symmetry which is the combination of $\hat{S}$ with a gauge symmetry $\hat{K}^{1/n}$ (which is also broken by $\langle\phi\rangle$), where $\hat{K}$ is a generator of the low energy $\mathbb{Z}_{\bar{k}}$ gauge group, and we denote it as

$$\hat{H} = \hat{S} \cdot \hat{K}^{1/n} = \exp(2\pi i(\bar{k}\hat{s} + \hat{k})/\bar{k}n) \in \mathbb{Z}_{\bar{k}n}, \tag{4.48}$$

where $\hat{s}$ and $\hat{k}$ are the charge operators associated with the $\mathbb{Z}_n$ and the U(1) symmetries, respectively. The definition of $\hat{H}$ is not unique; any operator having the form $\hat{H} \cdot \hat{K}'$ for any $\hat{K}' \in \mathbb{Z}_{\bar{k}}$ is also a symmetry of the low energy phase. The point is that $\hat{H}$ (or any $\hat{H} \cdot \hat{K}'$) has the following property

$$\hat{H}^n = \hat{S}^n \cdot \hat{K} \in \mathbb{Z}_{\bar{k}}, \tag{4.49}$$

so that any physical state that is invariant under the low energy $\mathbb{Z}_{\bar{k}}$ gauge symmetry satisfies $\hat{H}^n = 1$. On the other hand, individual gauge non-invariant quasiparticles transform under a $\mathbb{Z}_{\bar{k}n}$ symmetry, so the low energy gapped phase respects a global $\mathbb{Z}_n$ symmetry that is realized projectively in the presence of the $\mathbb{Z}_{\bar{k}}$ gauge symmetry

Therefore, if we are given a set of Weyl fermions $\{\psi_i\}$ with $\mathbb{Z}_n$ charges $\{s_i\}$ that obey the condition (4.46) or (4.47) for some $\bar{k} \in \mathbb{Z}$, it is possible, using the model presented here, to construct a gapped state of fermions that also preserves a global $\mathbb{Z}_n$ symmetry. If there exists a solution for (4.46) or (4.47) with $\bar{k} = 1$, such a symmetric gapped state is topologically trivial and can be realized in a purely 3+1-dimensional system. On the other hand, if one can only find a solution for (4.46) or (4.47) with some integer $\bar{k} \neq 1$, the corresponding gapped state would be topologically nontrivial, as there is a $\mathbb{Z}_{\bar{k}}$ gauge symmetry emergent at low energy and the (total) symmetry of the system is lifted from $\mathbb{Z}_n$ to $\mathbb{Z}_{\bar{k}n}$ by this $\mathbb{Z}_{\bar{k}}$ gauge symmetry; this gapped state is anomalous (associated with the $\mathbb{Z}_n$ global symmetry) and must be realized as a boundary state of a 4+1-dimensional fermionic $\mathbb{Z}_n$ SPT phase. Thus, the physical model of gapped boundary states via weak coupling we considered agrees with the analysis by purely geometrical considerations (that is, by knowledge from the groups $\Gamma^5_{\mathsf{Spin}}(B\mathbb{Z}_n)$) in the last section, as expected. In this model, however, we are not sure if there always exists a set of Weyl fermions $\{\chi_j\}$ in an anomaly-free U(1) representation such that (4.46) or (4.47) has a solution for each given $\{\psi_i\}$ with $\mathbb{Z}_n$ charges $\{s_i\}$ that satisfy this condition. This situation is the same as the Ibanez-Ross condition for an anomaly-free $\mathbb{Z}_n$ symmetry.

Examples

Let us look at some examples for the construction of gapped boundary states. Consider ν right-handed Weyl fermions $\{\psi_i\}$ in a representation $R = \oplus_{i=1}^{\nu}\rho_{s_i} = \oplus_{i=1}^{\nu}e^{2\pi i s_i/4}$ of a $\mathbb{Z}_4$ global symmetry. Since we have $\alpha_{\rho_2} = 0$ and $\alpha_{\rho_3} = -\alpha_{\rho_1}$ (without symmetry breaking, a Weyl fermion with $\mathbb{Z}_4$ charge 2 can be gapped by a Majorana mass, while a pair of Weyl fermions with $\mathbb{Z}_4$ charges 1 and 3 can be gapped by a Dirac mass), we can focus on the case where all $s_i = 1$:

(1) For $\nu = 0 \mod 4$, $\alpha_R = (\nu \mod 4,\ \nu - \nu \mod 2) = 0$ and thus the theory is anomaly-free. Let us take $\nu = 4$ for discussion. To construct a gapped, symmetry-preserving, and topological trivial state by the model above, we can take, for example, the U(1) charges of $\{\chi_j\}$ as $\{k_j\} = \{3, -5, -5, -5, -1, 5, 2, 6\}$, such that $\chi_{i'}$ and ψ_i for $i' = i = 1, ..., 4$, χ_5 and χ_6, and χ_7 and χ_8 are all paired with Dirac-type masses (there are no fermions with Majorana-type masses in this case), and also take the $\mathbb{Z}_4$ and the U(1) charges $\bar{s}$ and $\bar{k}$ of ϕ to be -1 and 1, respectively. The corresponding low energy phase in the presence of a nonzero $\langle\phi\rangle$ is invariant under a $\mathbb{Z}_4$ global symmetry $\hat{H} = \exp(2\pi i(\hat{s} + \hat{k})/4)$ that comes from a breakdown of the (high energy) $U(1) \times \mathbb{Z}_4$ symmetry.

(2) For $\nu = 2 \mod 4$, $\alpha_R = (2 \mod 4,\ 0 \mod 2) \neq 0$, while the linear $\mathbb{Z}_4$ anomaly $\sum_i s_i \mod 2$ vanishes. The theory is anomalous and must appear on the boundary of a five-dimensional SPT phase with $\mathbb{Z}_4$ symmetry. Take $\nu = 2$ for discussion. To realize an anomalous gapped boundary state by our model, we can take, for example, $\{k_j\} = \{-2, -10, 7, 9, -4\}$, such that χ_1 and ψ_1, χ_2 and ψ_2, and χ_3 and χ_4 are all paired with Dirac-type masses, while the fermion χ_5 is with a Majorana-type mass. In this case, $\bar{s} = -1$ and $\bar{k} = 2$, so the low energy phase has a $\mathbb{Z}_2$ gauge symmetry and we can define a $\mathbb{Z}_4$ global symmetry via a symmetry $\hat{H} = \exp(2\pi i(2\hat{s}+\hat{k})/8) \in \mathbb{Z}_8$, as any state of a compact sample satisfies $\hat{H}^4 = \exp(\pi i\hat{k}) = 1$. Note

that individual quasiparticles such as χ_3 and χ_4 have a "symmetry-fractionalization" relation $\hat{H}^4 = -1$.

(3) For odd ν, $\alpha_R \neq 0$ and the linear $\mathbb{Z}_4$ anomaly does not vanish. From the previous discussion, it is impossible to trivialize α_R through any (nontrivial) extensions of $\mathbb{Z}_4$ to $\mathbb{Z}_{4m}$, and thus we can not have a gapped (boundary) state from any model associated with this kind of extensions. Again, as mentioned previously, we are not sure if it is possible to find a trivialization of α_R via a nontrivial extension of $\mathbb{Z}_4$ to a more general group $\mathcal{H}$.

Chapter 5

Conclusions and outlook

In this thesis, we have tried to establish a theoretic framework for studying interacting SPT phases. Our approach, based mainly on the concept of anomalies, not only provides a fundamental viewpoint on understanding the physics behind these exotic phases, but also gives an efficient and elegant way to diagnose the interaction effects. We have studied topological phases protected by internal (such ascharge U(1) symmetry) and/or discrete spacetime (such as time reversal and spatial reflection) symmetries in two (chapter 2) and three (chapter 3) spatial dimensions. A nontrivial fact, unlike the usual procedure of gauging an internal symmetry, is one needs to consider the boundary theories of SPT phases with such spacetime symmetries on unorientable spacetime manifolds, which arise naturally from enforcing (or weakly gauging) these symmetries. As a check, our results by the anomaly argument agree with the ones by analyzing the stability (or ingappability) of the boundary theories upon the inclusion of many-body interactions.

For example, the $\mathbb{Z}_2$ classification of the (2+1)d topological insulator protected by charge U(1) and time-reversal (or CP) symmetries can be deduced by the form of the global U(1) gauge anomalies on its edge theories defined on closed unorientable manifolds. In this case, the nontrivial phase (in free systems) is robust against electron interactions. Another example is the (3+1)d topological superconductor protected by only time-reversal or reflection symmetry. For this system, we identified the bulk phase by studying the global gravitational anomalies on its surface theories, and also discussed the connection to the collapse of the non-interacting classification by an integer $\mathbb{Z}$ to $\mathbb{Z}_{16}$, in the presence of interactions.

From the perspective of fermionic SPT phases in 4+1 dimensions, we revisit the problem of gauging a discrete internal symmetry in theories of chiral (Weyl) fermions in 3+1 dimensions (in chapter 4), which has been studied in [60] 25 years ago. Comparing with the their results, we give a more complete solution for the anomalies constraints on the discrete symmetry ($\mathbb{Z}_n$ symmetry in particular), as our approach is based on purely geometrical considerations, namely, our assumption is more fundamental and general. Furthermore, our results also provides an understanding of gapped states of fermions with anomalous discrete symmetries, and we present a model, based on weak coupling, for constructing these anomalous gapped states.

Our works have made partial, but considerable progress in solving problems in many-body quantum systems, and can constitute a solid basis for future relevant studies. Our results are significant not only in condensed matter physics for the approaches to studying strongly correlated systems, but also in fundamental physics for the newly discovered formalism of anomalies.

It would be interesting to explore more generic interacting topological phases of matter in the future. There are several research goals along this line: Given a generic set of symmetries, we aim at

(1) knowing the complete classification of interacting SPT phases with these symmetries and also identifying the corresponding topological invariants;

(2) finding the underlying theories, given by either hydrodynamic (effective field theory) descriptions or microscopic constructions, that describe the nontrivial interacting SPT phases beyond free fermions;

(3) constructing various anomalous gapped boundary states (with exotic topological orders) of a given bulk SPT phase and investigating the phase structure of these states (such anomalous gapped boundary states can exist even in a bulk SPT phase consisting of free fermions);

(4) identifying the physical quantities that probe the topological character of the nontrivial SPT phases and can be measured (by a feasible experimental setup).

To understand the complete classification of SPT phases with a given symmetry group, we need to classify the corresponding boundary anomalies. There are two types of anomalies, the perturbative and global (or non-perturbative) ones. The forms of perturbative anomalies do not depend on the manifolds on which theories are formulated and have been reasonably studied within the framework of quantum field theory. On the other hand, global anomalies do depend on the topology of the manifolds (equipped with some additional symmetry structures), and until very recently their definition has been reformulated in a more consistent way within the framework of free fermion theory and topological quantum field theories [56]. It has been proposed that global anomalies can be classified by the cobordism theory [88]. Therefore, the most challenging part of classifying SPT phases with a generic symmetry group (in arbitrary dimensions) is to find the cobordism group with the same symmetry group, which is one of the main directions for my future works.

Once the associated cobordism group is known, the topological (cobordism) invariants, interpreted as the effective actions of bulk SPT phases, are also determined. While some of them can be identified as the quantum partition functions of some free field theories (in the low energy limit), the others that are constructed by some specific characteristic numbers seem "purely topological" and might not have expressions in terms of local degrees of freedom. By this observation, I can identify the bulk phases which are originated from free (or weakly interacting) fermions, while the other phases are known to be strongly interacting (some of these phases might have effective field theory descriptions). One of my research goals is to present physical models, in a systematic way, that realize these strongly interacting topological phases and also to understand the underlying mechanism, regarding the interaction effects, that drives the quantum phase transitions (that separate difference bulk phases) occurring in the bulk.

On the other hand, for a given bulk SPT phase (in the topological limit), there may be multiple boundary theories with distinct spectra and dynamics — a many-to- one bulk-boundary correspondence — as they possess the same form of (boundary) anomalies. There are a number of ways to construct such anomalous boundary states, as one starts from a "standard" state such as a gapless free fermion phase. In two spatial dimensions, one can use either the vortex condensation approach [30] or the Higgs mechanism in weakly coupled theories [135] to engineer various exotic topological orders on the surfaces of (3+1)d topological insulators and superconductors. It is worth studying to apply these methods to construct such anomalous gapped boundary states in generic SPT phases (in three or higher spatial dimensions).

Finally, a practical issue naturally arises: is it always possible to"measure" the topological nature of a generic interacting topological phase? For quantum Hall systems, one can measure the quantized Hall conductances in the presence of electromagnetic background fields. For the three-dimensional topological insulator, the $\mathbb{Z}_2$ index of the nontrivial phase can be detected by the so-called magnetoelectric effect, as described by axion electrodynamics [136]. For generic topological phases that manifest (boundary) perturbative anomalies (gauge and/or gravitational ones), one can diagnose the bulk topological nature by electromagnetic or thermal (or others regarding the underlying symmetries) responses, regardless of the interparticle interactions [37]. For SPT phases that possess only (boundary) global anomalies, this question becomes more subtle, and in general, it is not clear whether we can always formulate (a similar kind of) response theories to extract the physical quantities that probe the nontrivial bulk topologies (especially for those topological indices other than $\mathbb{Z}_2$). A possible direction is to consider a generalized Laughlin's argument when looking into such SPT phases, basing on the idea in our previous work [54] (or chapter 2). It would be very interesting and also significant to make the connection to experimental realizations.

Appendix A

A.1 The CP eigenvalue of the ground state

In the bulk of the paper, we have enforced CP invariance at all steps of an adiabatic evolution (for all values of the flux a). In fact, the system (defined by Lagrangian with boundary conditions) is classically CP invariant, and hence one would assume this is so even at quantum level. What we discovered, under this assumption, is the violation of the electromagnetic U(1) symmetry. As we mentioned in Sec. 2.2.2, an alternative point of view is possible; if we were to enforce the electromagnetic U(1) symmetry, CP would then be violated (when the CP symmetry in question is "topological" – the one which leads topological insulators). Therefore, while the system preserves, at the classical level, both the electromagnetic U(1) and CP symmetries, there is a tension between these symmetries once we quantize the system. Once we demand the electromagnetic U(1) symmetry be strictly conserved, instead of enforcing CP, $P_{[a]}$ may not be independent of a. In the following, we determine $P_{[a]}$ under the assumption of the U(1) conservation.

The IQHE As a warm up, let us start from an edge state of the (integer) quantum Hall system; it suffers from an anomaly, and hence cannot exist on its own. (We follow closely Ref. [137]). The edge state of the IQHE is a chiral fermion ψ_R. We consider an edge of circumference 2π, and impose the twisted boundary condition: $\psi_R(x+2\pi) = e^{2\pi i\nu}\psi_R(x)$. From the state-operator correspondence, there is an operator associated to the ground state for a given ν, which we call $\mathcal{A}_\nu$. The operator can be determined from the following general principle: (i) any unitary on-site symmetry in field theories can be used to generate a twisting boundary condition; (ii) in CFT, Hilbert space with twisted boundary condition form an independent sector (Virasoro module); (iii) due to the state-operator correspondence, there is an operator that corresponds to a ground state of the twisted Hilbert space. The identification of the ground state operator can be done conveniently in terms of bosonization:

$$\psi_R \simeq e^{i\varphi_R}. \tag{A.1}$$

One could then infer the operator corresponding to the ground state:

$$\mathcal{A}_\nu \equiv e^{i(-\nu+1/2)\varphi_R}, \tag{A.2}$$

where φ_R is a chiral boson field. From this expression for the ground state operator one infers that the charge of the ground state is

$$F_R = 1/2 - \nu. \tag{A.3}$$

Thus, we conclude that the ground state fermion number at the edge of the quantum Hall system changes as a function of twisting angle. Because of the spectral flow, as one changes $a \to a+1$, the fermion number jumps by one (discontinuously). Had the charge been conserved (*i.e.*, had there been no anomaly), the ground state fermion number should be independent of the twisting angle. The ground state charge (A.3) is the origin of the factor $e^{-2\pi i(b-1/2)(a-1/2)}$ in the partition function (2.3) [1].

The QSHE with conserved S_z Let us now consider the edge theory of a bulk quantum spin Hall system with conserving S_z. The edge state now consists of both left- and right-movers, ψ_L and ψ_R. These fermion fields can be bosonized as

$$\psi_L \sim e^{i\varphi_L}, \quad \psi_R \sim e^{-i\varphi_R}. \tag{A.4}$$

(Here, we do not include Klein factors while they are important in discussing CP symmetric topological insulators.) Following the same argument as in the case of the QHE, the operator corresponding the ground state of the left-moving sector is $e^{i(-\nu_L+1/2)\varphi_L}$ where ν_L is the twisting angle for the left movers. Similarly, the ground state for the right moving sector can be represented as $e^{i(-\nu_R+1/2)\varphi_R}$ By combining the left- and right-moving parts of the ground state properly, we have a ground state for the combined non-chiral system.

[1] While we have determined the ground state and its charge as above, we could take an alternative point of view. Let us assume that we actually do not know, a priori, that the U(1) symmetry is anomalous. We would like to test if this symmetry is anomalous or not. For this purpose, we pretend the charge U(1) is conserved. We do so since the charge U(1) is classically conserved, and if so, one would guess naively that the ground state fermion number does not change as we adiabatically change a and b. Under this assumption, what one would discover is that the partition function is not invariant under $a \to a+1$. Therefore, even though we started from the assumption that the U(1) is conserved, we run into the "inconsistency" in that the partition function is not invariant under $a \to a+1$, in stead of $b \to b+1$ – we then conclude we cannot conserve the U(1) at the quantum level.

Below, we put more emphasis on U(1) charge conservation than S_z conservation; we first give a priority to the U(1) charge conservation and see this necessary leads to violation of S_z conservation – this is nothing but chiral anomaly. Since charge U(1) is conserved, it makes sense to twist boundary conditions by charge U(1) symmetry, $\nu_L = \nu_R$. We are thus lead to the ground state vertex operator

$$e^{i(-\nu+1/2)\varphi_L} e^{+i(-\nu+1/2)\varphi_R} = e^{i(-\nu+1/2)\phi}. \tag{A.5}$$

Here, the non-chiral field $\phi = \varphi_L + \varphi_R$ is charge neutral. One could combine the left- and right-moving sectors differently to get $e^{i(-\nu+1/2)\theta}$ with $\theta = \varphi_L - \varphi_R$. This choice, however, is not consistent with charge U(1) conservation since θ is not charge neutral and the ground state fermion number $F_V = F_L + F_R$ changes as a function of ν, $\nu = \nu + 1/2$.

While the ground state $e^{i(-\nu+1/2)\phi}$ is consistent with charge U(1) conservation, the price we paid is that the ground state is charged under spin S_z conservation. This means that as one adiabatically inserts charge flux, the S_z quantum number of the edge state changes – the spin is "pumped" from the edge in question to other edges, or vice versa.

CP symmetric bosonic topological insulators Let us now break the continuous the U(1) spin S_z conservation and instead impose CP symmetry; we consider the case of CP symmetric topological insulators. The relevant symmetries are charge U(1) and CP. In particular, we focus on the bosonic version of the topological insulator that we discuss in Sec. 2.3. The CP acts on the bosonic field as in Eq. (2.42). Following above discussion, we consider the ground state that preserves the electromagnetic U(1) symmetry as a function of twisting angle ν. The ground state is then given by

$$e^{i\nu\theta R/\alpha'} \tag{A.6}$$

The CP eigenvalue of the ground state is

$$(\mathcal{CP}) e^{i\nu\theta R/\alpha'} (\mathcal{CP})^{-1} = P_{[\nu]} e^{i\nu\theta R/\alpha'}, \text{ where } \quad P_{[\nu]} = e^{i2\pi\nu\epsilon}. \tag{A.7}$$

Thus, for the topologically trivial case $\epsilon = 0$, the CP eigenvalue is independent of ϵ, where as when $\epsilon = 1/2$ (topological), the ground state CP eigenvalue evolves as a function of ν. This signals the conflict of the symmetry; once we choose to preserve the U(1), CP is necessarily broken.

A.2 Statistical phase factor of the chiral boson field under symmetry transformation

For any local quasiparticle excitation $\ddagger \exp i\mathbf{\Lambda}^T K\boldsymbol{\phi}\ddagger$, where $\mathbf{\Lambda}^T K\boldsymbol{\phi} = \sum_I \Lambda_I (K\boldsymbol{\phi})_I \equiv \sum_I \theta_I$, the symmetry transformation $\mathcal{G}$ acts as

$$\begin{aligned}
&\mathcal{G} \ddagger e^{i\mathbf{\Lambda}^T K\boldsymbol{\phi}} \ddagger \mathcal{G}^{-1} = \mathcal{G} \ddagger e^{\sum_J i\theta_I} \ddagger \mathcal{G}^{-1} \\
&= \mathcal{G} \ddagger \prod_I{}' e^{i\theta_I} \cdot e^{-\frac{1}{2}\sum_{I<J}[i\theta_I, i\theta_J]} \ddagger \mathcal{G}^{-1} \\
&= \ddagger \prod_I{}' e^{\mathcal{G}i\theta_I\mathcal{G}^{-1}} \ddagger \cdot e^{-\frac{1}{2}\sum_{I<J}\mathcal{G}[i\theta_I, i\theta_J]\mathcal{G}^{-1}} \\
&\equiv \ddagger \prod_I{}' e^{\mathcal{G}i\theta_I\mathcal{G}^{-1}} \ddagger \cdot e^{-\frac{1}{2}\sum_{I<J}[\mathcal{G}i\theta_I\mathcal{G}^{-1}, \mathcal{G}i\theta_J\mathcal{G}^{-1}]} \cdot e^{i\Delta\phi^{\mathbf{\Lambda}}_{\mathrm{G}}} \\
&= \ddagger e^{\sum_j \mathcal{G}i\theta_I\mathcal{G}^{-1} + i\Delta\phi^{\mathbf{\Lambda}}_{\mathrm{G}}} \ddagger,
\end{aligned} \tag{A.8}$$

where we have used the Baker-Campbell-Hausdorff formula (with the commutator $[i\theta_I, i\theta_J]$ being a c-number), the ordered-product "$\prod_I'$" is defined as an ordered product in the ascending order of indices, and

$$i\Delta\phi^{\mathbf{\Lambda}}_{\mathrm{G}} \equiv \frac{1}{2}\sum_{I<J}\left([\mathcal{G}i\theta_I\mathcal{G}^{-1}, \mathcal{G}i\theta_J\mathcal{G}^{-1}] - \mathcal{G}[i\theta_I, i\theta_J]\mathcal{G}^{-1}\right). \tag{A.9}$$

Note that we keep the form $\mathcal{G}[i\theta_I, i\theta_J]\mathcal{G}^{-1}$ even if $[i\theta_I, i\theta_J]$ is a c-number, since in general $\mathcal{G}$ can be an antiunitary operator (*e.g.* T symmetry). On the other hand,

$$\mathcal{G}e^{i\mathbf{\Lambda}^T K\boldsymbol{\phi}}\mathcal{G}^{-1} = e^{\mathcal{G}i\mathbf{\Lambda}^T K\boldsymbol{\phi}\mathcal{G}^{-1}} = e^{\mathcal{G}\left(\sum_I i\theta_I\right)\mathcal{G}^{-1}}, \tag{A.10}$$

so we have

$$\mathcal{G}\left(i\mathbf{\Lambda}^T K\boldsymbol{\phi}\right)\mathcal{G}^{-1} = \sum_I \mathcal{G}i\theta_I\mathcal{G}^{-1} + i\Delta\phi^{\mathbf{\Lambda}}_{\mathrm{G}} \mod 2\pi i. \tag{A.11}$$

This means the way that the operator $\mathcal{G}$ acts on the chiral boson field ϕ is not always linear, because some nontrivial phase factor $\Delta\phi^{\mathbf{\Lambda}}_{\mathrm{G}}$ ($\neq 2n\pi$) might arise. In bosonic system, the phase factor is always the multiple of $2\pi i$, corresponding to Bose statistics, and thus we can ignore it (in this case $\mathcal{G}$ is linear in ϕ). In fermionic systems, however, we must be careful with the phase factor, which might be nontrivial, because of the Fermi statistics.

In the following we take CP and T symmetries as examples.

CP symmetry From the canonical commutation relation (2.66), when $x \neq x'$ (but $x \to x'$ is taken when we consider the operator product expansion of vertex operators) and $I \neq J$, we have

$$[(K\boldsymbol{\phi})_I(t,x),(K\boldsymbol{\phi})_J(t,x')] = -i\pi\mathrm{sgn}(I-J)Q_IQ_J + 2\pi i N_{IJ}, \tag{A.12}$$

where N_{IJ} is the component of an integer matrix. Now for CP symmetry defined in Sec. 2.4.2, the extra phase is given by

$$i\Delta\phi_{\mathrm{CP}}^{\boldsymbol{\Lambda}} = -\frac{1}{2}\sum_{I<J}^{N}\Lambda_I\Lambda_J\Big\{[(U_{\mathrm{CP}}K\boldsymbol{\phi})_I,(U_{\mathrm{CP}}K\boldsymbol{\phi})_J] - [(K\boldsymbol{\phi})_I,(K\boldsymbol{\phi})_J]\Big\}. \tag{A.13}$$

Since U_{CP} has the form $\left(\begin{smallmatrix} 0 & I_{N/2} \\ I_{N/2} & 0 \end{smallmatrix}\right)$, where N is an even integer, we have, for $1 \leq I < J \leq N$,

$$\begin{aligned} &[(U_{\mathrm{CP}}K\boldsymbol{\phi})_I,(U_{\mathrm{CP}}K\boldsymbol{\phi})_J] \\ &= \begin{cases} +i\pi(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J & \text{if } 1 \leq I < J \leq N/2 \\ & \text{or } N/2+1 \leq I < J \leq N \\ -i\pi(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J & \text{if } 1 \leq I \leq N/2 \\ & \text{and } N/2+1 \leq J \leq N \end{cases} \\ &\quad \bmod 2\pi i. \end{aligned} \tag{A.14}$$

Then

$$
\begin{aligned}
i\Delta\phi_{\mathrm{CP}}^{\boldsymbol{\Lambda}} = & -\frac{i\pi}{2}\sum_{1\le I<J\le N/2}\Lambda_I\Lambda_J\left[(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J-Q_IQ_J\right] \\
& -\frac{i\pi}{2}\sum_{N/2+1\le I<J\le N}\Lambda_I\Lambda_J\left[(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J-Q_IQ_J\right] \\
& +\frac{i\pi}{2}\sum_{\substack{1\le I\le N/2\\ N/2+1\le J\le N}} i\Lambda_I\Lambda_J\left[(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J+Q_IQ_J\right] \\
= & \frac{i\pi}{2}\sum_{\substack{1\le I\le N/2\\ N/2+1\le J\le N}}\Lambda_I\Lambda_J\left[(U_{\mathrm{CP}}\boldsymbol{Q})_I(U_{\mathrm{CP}}\boldsymbol{Q})_J+Q_IQ_J\right] \\
= & \, i\pi\left(\sum_{I=1}^{N/2}\Lambda_IQ_I\right)\left(\sum_{J=N/2+1}^{N}\Lambda_JQ_J\right) \mod 2\pi i,
\end{aligned}
\tag{A.15}
$$

where the second equality holds since the sum of the first two terms in the first equality vanishes. For CP invariant vector $\boldsymbol{\Lambda}$, with $\boldsymbol{\Lambda}=-U_{\mathrm{CP}}\boldsymbol{\Lambda}$, we can express $\boldsymbol{\Lambda}$ as $(\boldsymbol{\lambda},-\boldsymbol{\lambda})^T$, where $\boldsymbol{\lambda}$ is any $N/2$ dimensional integer vector. Then the statistical phase can be expressed as

$$
i\Delta\phi_{\mathrm{CP}}^{\boldsymbol{\Lambda}} = -\, i\pi\left(\sum_{I=1}^{N/2}\lambda_Iq_I\right)^2 = -i\pi\sum_{I=1}^{N/2}\lambda_Iq_I = -i\pi\boldsymbol{\lambda}^T\boldsymbol{q} \mod 2\pi i. \tag{A.16}
$$

T symmetry The set of data $\{K,\boldsymbol{Q},U_{\mathrm{T}},\boldsymbol{\chi}_{\mathrm{T}}\}$ for the T symmetric K-matrix theory is the same as the case of CP. The only difference is that T is an antiunitary operator, which results in [from Eq. (A.12)]

$$
\mathcal{T}[(K\boldsymbol{\phi})_I,(K\boldsymbol{\phi})_J]\mathcal{T}^{-1} = -[(K\boldsymbol{\phi})_I,(K\boldsymbol{\phi})_J], \tag{A.17}
$$

and hence reverses the sign in front of $Q_I Q_J$ in Eq. (A.15)

$$\begin{aligned} i\Delta\phi_{\mathrm{T}}^{\boldsymbol{\Lambda}} &= -\frac{i\pi}{2}\sum_{1\le I<J\le N/2}\Lambda_I\Lambda_J\left[(U_{\mathrm{T}}\boldsymbol{Q})_I(U_{\mathrm{T}}\boldsymbol{Q})_J + Q_IQ_J\right] \\ &\quad -\frac{i\pi}{2}\sum_{N/2+1\le I<J\le N}\Lambda_I\Lambda_J\left[(U_{\mathrm{T}}\boldsymbol{Q})_I(U_{\mathrm{T}}\boldsymbol{Q})_J + Q_IQ_J\right] \\ &\quad +\frac{i\pi}{2}\sum_{\substack{1\le I\le N/2\\ N/2+1\le J\le N}}\Lambda_I\Lambda_J\left[(U_{\mathrm{T}}\boldsymbol{Q})_I(U_{\mathrm{T}}\boldsymbol{Q})_J - Q_IQ_J\right] \\ &= -i\pi\left(\sum_{1\le I<J\le N/2} + \sum_{N/2+1\le I<J\le N}\right)\Lambda_I\Lambda_JQ_IQ_J \\ &\quad \mod 2\pi i, \end{aligned} \tag{A.18}$$

where the second equality holds since the third term in the first equality vanishes. For a T-invariant vector $\boldsymbol{\Lambda}$ satisfying $\boldsymbol{\Lambda} = -U_T\boldsymbol{\Lambda}$, we can express $\boldsymbol{\Lambda}$ as $(\boldsymbol{\lambda}, -\boldsymbol{\lambda})^T$, where $\boldsymbol{\lambda}$ is any $N/2$ dimensional integer vector. Then the statistical phase is

$$i\Delta\phi_{\mathrm{T}}^{\boldsymbol{\Lambda}} = -2\pi i\sum_{1\le I<J\le N/2}\lambda_I\lambda_Jq_Iq_J = 0 \mod 2\pi i. \tag{A.19}$$

Therefore, when discussing the K-matrix theory with T symmetry, statistical phases are irrelevant and can safely be ignored, as pointed out in Ref. [90].

Appendix B

B.1 The Dirac fermion theory on two-torus T^2

In this appendix, we review the modular invariance, the $\mathrm{SL}(2,\mathbb{Z})$ invariance, of the Dirac fermion theory on two-torus T^2.

For a flat T^2, the zweibein can be factorized as

$$e^A{}_\mu = \begin{pmatrix} R_0 & 0 \\ 0 & R_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} R_0 & 0 \\ -\alpha R_1 & R_1 \end{pmatrix}, \tag{B.1}$$

and its inverse is given by

$$e^{\star}_A{}^\mu = \begin{pmatrix} \frac{1}{R_0} & \frac{\alpha}{R_0} \\ 0 & \frac{1}{R_1} \end{pmatrix}, \tag{B.2}$$

such that $e^A{}_\mu e^{\star}_A{}^\nu = \delta_\mu{}^\nu$ and $e^A{}_\mu e^{\star}_B{}^\mu = \delta^A{}_B$. Here R_0 and R_1 are the radii for the directions 0 and 1, and α is related to the angle between the directions 0 and 1. The Euclidean metric is then given by

$$g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \delta_{AB} = \begin{pmatrix} R_0^2 + \alpha^2 R_1^2 & -\alpha R_1^2 \\ -\alpha R_1^2 & R_1^2 \end{pmatrix}, \tag{B.3}$$

and the corresponding line element is

$$ds^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu = R_0^2 (d\theta^0)^2 + R_1^2 (d\theta^1 - \alpha d\theta^0)^2, \tag{B.4}$$

where $0 \leq \theta^\mu \leq 2\pi$ are angular variables

The group $\mathrm{SL}(2,\mathbb{Z})$ is generated by two transformations:

$$U_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.5}$$

$\mathrm{SL}(2,\mathbb{Z})$ transformations on the zweibein and metric are induced by

$$\begin{aligned} e^A{}_\mu &\xrightarrow{L} \left(eL^T\right)^A{}_\mu = L_\mu{}^\rho e^A{}_\rho, \\ e^{\star}{}_A{}^\mu &\xrightarrow{L} \left(e^\star L^{-1}\right)_A{}^\mu = e^{\star}{}_A{}^\rho \left(L^{-1}\right)_\rho{}^\mu, \\ g_{\mu\nu} &\xrightarrow{L} \left(LgL^T\right)_{\mu\nu} = L_\mu{}^\rho L_\nu{}^\sigma g_{\rho\sigma}, \end{aligned} \tag{B.6}$$

for any $\mathrm{SL}(2,\mathbb{Z})$ element $L = U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$. In particular,

$$g_{\mu\nu} \xrightarrow{U_1} \left(U_1 g U_1^T\right)_{\mu\nu} = \begin{pmatrix} R_1^2 & \alpha R_1^2 \\ \alpha R_1^2 & R_0^2 + \alpha^2 R_1^2 \end{pmatrix}, \tag{B.7}$$

which corresponds to the changes

$$R_0 \to R_0/|\tau_{2d}|, \quad R_1 \to R_1|\tau_{2d}|, \quad \alpha \to -\alpha/|\tau_{2d}|^2, \tag{B.8}$$

or, in terms of the modular parameter (the Teichmüller parameter) $\tau_{2d} \equiv \alpha + i\frac{R_0}{R_1}$,

$$\tau_{2d} \to -1/\tau_{2d}. \tag{B.9}$$

On the other hand,

$$g_{\mu\nu} \xrightarrow{U_2} \left(U_2 g U_2^T\right)_{\mu\nu} = \begin{pmatrix} R_0^2 + (\alpha-1)^2 R_1^2 & -(\alpha-1)R_1^2 \\ -(\alpha-1)R_1^2 & R_1^2 \end{pmatrix}, \tag{B.10}$$

which corresponds to the change

$$\alpha \to \alpha - 1 \quad \text{(while } R_0 \text{ and } R_1 \text{ are unchanged)}. \tag{B.11}$$

The two transformations U_1 and U_2 are exactly S and T^{-1} transformations that generate $\mathrm{SL}(2,\mathbb{Z})$ (usually used in the 2d conformal field theory literatures), respectively.

The Euclidean action for the Dirac fermion on this two torus is given by

$$S_E = \frac{1}{2\pi}\int d^2\theta \ \ (\det e)\, \bar{\psi}\left(\Gamma^A e^{\star\ \mu}_{A}\partial_{\theta^\mu}\right)\psi, \tag{B.12}$$

where $\det e = \sqrt{g} = R_0 R_1$, $\partial_{\theta^\mu} \equiv \frac{\partial}{\partial\theta^\mu}$, and the gamma matrices Γ^A satisfy $\{\Gamma^A, \Gamma^B\} = 2\delta^{AB}$. In terms of the space-time coordinates $\tau = R_0\theta^0$ and $x = R_1\theta^1$:

$$\begin{aligned} 2\pi S_E &= \int_0^{2\pi} d\theta^0 \int_0^{2\pi} d\theta^1 \bar{\psi}\left(R_1\Gamma^0\partial_{\theta^0} + \alpha R_1\Gamma^0\partial_{\theta^1} + R_0\Gamma^1\partial_{\theta^1}\right)\psi \\ &= \int_0^{2\pi R_0} d\tau \int_0^{2\pi R_1} dx \bar{\psi}\left(\Gamma^0\partial_\tau + \alpha\frac{R_1}{R_0}\Gamma^0\partial_x + \Gamma^1\partial_x\right)\psi. \end{aligned} \tag{B.13}$$

The partition function can be evaluated by the path integral on the (general) two torus $Z(g) = \int \mathcal{D}[\psi^\dagger, \psi]e^{-S_E}$, or by the operator formalism

$$Z(g) = \mathrm{Tr}\left[e^{-2\pi R_0 H'}\right], \tag{B.14}$$

where H' is the "boosted" Hamiltonian (in the presence of non-vanishing α) corresponding to S_E:

$$H' = H - i\alpha\frac{R_1}{R_0}P_x, \tag{B.15}$$

with

$$\begin{aligned} H &= \frac{1}{2\pi}\int_0^{2\pi R_1} dx\, \bar{\psi}\Gamma^1\partial_x\psi, \\ P_x &= \frac{1}{2\pi}\int_0^{2\pi R_1} dx\, \psi^\dagger(-i\partial_x\psi) \end{aligned} \tag{B.16}$$

being the Hamiltonian and momentum on a "flat two torus" ($\alpha = 0$).

The modular invariance for the partition function of nonchiral fermions is achieved by summing twisted partition functions over the spin structures. We thus consider the partition function

$$\mathcal{Z}^{tot}(g) = \sum_{(G_\tau, G_x)\in SG^2} \mathrm{Tr}_{G_x}\left[G_\tau(-1)^F e^{-2\pi R_0 H'}\right], \tag{B.17}$$

where $SG = \{1, (-1)^F\}$ is the symmetry group of the free fermion theory. Then, the total partition function satisfies $\mathcal{Z}^{tot}(LgL^T) = \mathcal{Z}^{tot}(g)$ for $L \in \mathrm{SL}(2, \mathbb{Z})$ [96].

B.2 Regularization of the ground state-energy

In this appendix, we regularize the ground-state energy, which is given by the divergent sum

$$E_{\mathrm{GS}[\mathsf{a}]}(g) = -\sum_{\mathsf{s}\in\mathbb{Z}^2+(a_x,a_y)} |\mathsf{s}|, \tag{B.18}$$

where $|\mathsf{s}| \equiv \sqrt{g_2^{ij}s_is_j}$.

Following Appendix C in Ref. [118], for arbitrary positive integer d, we have

$$\sum_{\mathsf{s}\in\mathbb{Z}^d+\alpha} |\mathsf{s}|e^{i\mathsf{s}\cdot\mathsf{x}} = \frac{c_{d+1}}{(2\pi)^d}\sqrt{g_d}\int d^dy \frac{1}{|\mathsf{y}|^{d+1}}\sum_{\mathsf{s}} e^{i\mathsf{s}\cdot(\mathsf{x}-\mathsf{y})}, \tag{B.19}$$

where $\mathsf{s},\alpha\in\mathbb{R}^d$, $|\mathsf{s}|\equiv\sqrt{g_d^{ij}s_is_j}$, $g_d\equiv\det(g_{dij})$, and $c_{d+1}\equiv\frac{\pi^{\frac{d}{2}}2^{d+1}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$. Now we use the equality

$$\sum_{\mathsf{m}\in\mathbb{Z}^d} e^{i\mathsf{m}\cdot(\mathsf{x}-\mathsf{y})} = (2\pi)^d\sum_{\mathsf{n}\in\mathbb{Z}^d}\delta^d(\mathsf{x}-\mathsf{y}+2\pi\mathsf{n}). \tag{B.20}$$

Substituting the above equality, with removing the $\mathsf{n}=0$ term in the sum, into (B.19), we obtain the regularized sum

$$\sum_{\mathsf{s}\in\mathbb{Z}^d+\alpha} |\mathsf{s}|e^{i\mathsf{s}\cdot\mathsf{x}} = c_{d+1}\sqrt{g_d}\int d^dy\frac{1}{|\mathsf{y}|^{d+1}}\sum_{\mathsf{n}\neq 0\in\mathbb{Z}^d}\delta^d(\mathsf{x}-\mathsf{y}+2\pi\mathsf{n})e^{i\alpha\cdot(\mathsf{x}-\mathsf{y})} = c_{d+1}\sqrt{g_d}\sum_{\mathsf{n}\neq 0\in\mathbb{Z}^d}\frac{e^{-2\pi i\alpha\cdot\mathsf{n}}}{|\mathsf{x}+2\pi\mathsf{n}|^{d+1}} \tag{B.21}$$

Then our regularized ground-state energy is given by

$$E_{\mathrm{GS}[\alpha]}(g) = -\sum_{\mathsf{s}\in\mathbb{Z}^2+(a_x,a_y)}|\mathsf{s}|\left.e^{i\mathsf{s}\cdot\mathsf{x}}\right|_{\mathsf{x}=0} = -c_3\sqrt{g_2}\sum_{\mathsf{n}\neq0\in\mathbb{Z}^2}\frac{e^{-2\pi i a^i n_i}}{|2\pi\mathsf{n}|^3} = \frac{1}{4\pi^2}\sqrt{\det(g_{2ij})}\sum_{\mathsf{n}\neq0\in\mathbb{Z}^2}\frac{\cos\left(2\pi i a^i n_i\right)}{\left(g_2^{ij}n_in_j\right)^{\frac{3}{2}}}. \tag{B.22}$$

B.3 Derivation of the claim (3.36)

In this Appendix, we confirm the claim (3.36) by explicitly checking how $Z_{[\mathsf{a}]}(g)$ transforms under the two generators $U_1 = U_1'M$ and U_2 of $\mathrm{SL}(3,\mathbb{Z})$, defined in Eqs. (3.9) and (3.13).

The behavior of $Z_{[\mathsf{a}]}(g)$ under U_2 and U_1' can be directly deduced by the properties of the massive theta function listed in (3.5).

Transformation under U_2: Under U_2, the metric is transformed as in (3.12), while the fluxes are transformed as $(a_\tau, a_x, a_y) \to (a_\tau + a_x, a_x, a_y)$. From (3.35), we have

$$\begin{aligned} Z_{[U_2\mathsf{a}]}(U_2 g {U_2}^T) &= \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[a_x+\beta s_y, a_\tau + a_x + (\gamma+\beta)s_y]}\left(\tau_{2d} - 1; r_{12} s_y\right) \\ &= \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[a_x+\beta s_y, a_\tau + \gamma s_y]}\left(\tau_{2d}; r_{12} s_y\right) \\ &= Z_{[\mathsf{a}]}(g) \end{aligned} \tag{B.23}$$

Transformation under U_1': Under U_1', the metric is transformed as in (3.15), while the fluxes are transformed as $(a_\tau, a_x, a_y) \to (-a_x, a_\tau, a_y)$. From (3.35), we have

$$\begin{aligned} Z_{[U_1'\mathsf{a}]}(U_1' g {U_1'}^T) &= \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[a_\tau+\gamma s_y, -a_x - \beta s_y]}\left(-1/\tau_{2d}; r_{12} s_y |\tau_{2d}|\right) \\ &= \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[-a_x-\beta s_y, -a_\tau - \gamma s_y]}\left(\tau_{2d}; r_{12} s_y\right) \\ &= \prod_{s_y \in \mathbb{Z}+a_y} \Theta_{[a_x+\beta s_y, a_\tau + \gamma s_y]}\left(\tau_{2d}; r_{12} s_y\right) \\ &= Z_{[\mathsf{a}]}(g) \end{aligned} \tag{B.24}$$

Transformation under M: Transformation for the parameters $\{R_i, \alpha, \beta, \gamma\}$ under M is not as obvious as the cases of U_2 and U_1'. We observe that, since the transformation M only involves the change in the x-y plane, under M the x- and y- components of the dreibein ${e^A}_\mu$ and the metric $g_{\mu\nu}$ (and their inverses) transform as:

$$\begin{aligned} {e^A}_i \to {M_i}^k {e^A}_k, \quad {e^\star_A}^i \to {e^\star_A}^k {(M^{-1})_k}^i, \\ g_{ij} \to {M_i}^k {M_j}^l g_{kl}, \quad (g_2)^{ij} \to {(M^{-1})_k}^i {(M^{-1})_l}^j (g_2)^{kl}, \end{aligned} \tag{B.25}$$

where $i, j, k, l = 1, 2$, and $(g_2)^{ij}$ is defined in Eq. (3.28). To see the behavior of Eq. (3.34) under M, we first note the regularized ground state energy (3.31) satisfies $E_{\mathrm{GS}[M\mathsf{a}]}(M g M^T) = E_{\mathrm{GS}[\mathsf{a}]}(g)$. On the other hand, the second line in Eq. (3.34) can be expressed as $(i, j = 1, 2)$

$$e^{-2\pi R_0 \varepsilon(\mathsf{s}) + 2\pi i \alpha s_x + 2\pi i(\alpha\beta+\gamma)s_y + 2\pi i a_\tau} = e^{-2\pi R_0 \sqrt{g_2^{ij} s_i s_j} + 2\pi i R_0 {e_0^\star}^i s_i + 2\pi i a_\tau}, \tag{B.26}$$

where $e_0^{\star i} = (\alpha/R_0, (\alpha\beta+\gamma)/R_0)^T$. From this expression, we can see that the mode-product term in Eq. (3.34) is also invariant under $\{g, \mathsf{a}\} \to \{MgM^T, M\mathsf{a}\}$. Therefore, we have shown

$$Z_{[M\mathsf{a}]}(MgM^T) = Z_{[\mathsf{a}]}(g). \tag{B.27}$$

From the above discussion, we thus confirm our claim (3.36).

B.4 Parity twisted partition functions of the surface theory of crystalline topological superconductors

In this Appendix, we explicitly calculate the partition functions twisted by parity, which is defined by

$$\mathscr{P}\psi(x,y)\mathscr{P}^{-1} = \sigma_3\psi(x,-y), \quad \mathscr{P}^2 = 1, \tag{B.28}$$

where ψ is the two-component Dirac fermion. (Remember that we have doubled the degree of freedom of the original theory of Majorana fermions.) Here we define $y \to -y$ instead $y \to 2\pi R_2 - y$ (defined in the main text) by parity is just for convenience (the result does not depend on the choice). As mentioned in the text, the parity invariance $\mathscr{P}H'\mathscr{P}^{-1} = H'$ forces strictly $\beta = \gamma = 0$. Then, P acts on the Fourier components of the original fermion operators as

$$\mathscr{P}\tilde{\psi}(\mathsf{s})\mathscr{P}^{-1} = \sigma_3\tilde{\psi}(\bar{\mathsf{s}}), \tag{B.29}$$

where $\bar{\mathsf{s}} = (s_x, -s_y)$. On the other hand, the P action on the eigen basis $\chi_\pm$, defined in Eq. (3.29), is deduced as

$$\mathscr{P}\chi(\mathsf{s})\mathscr{P}^{-1} = \begin{bmatrix} \langle u_+(\mathsf{s})|\sigma_3|u_+(\bar{\mathsf{s}})\rangle & \langle u_+(\mathsf{s})|\sigma_3|u_-(\bar{\mathsf{s}})\rangle \\ \langle u_-(\mathsf{s})|\sigma_3|u_+(\bar{\mathsf{s}})\rangle & \langle u_-(\mathsf{s})|\sigma_3|u_-(\bar{\mathsf{s}})\rangle \end{bmatrix} \chi(\bar{\mathsf{s}}). \tag{B.30}$$

where $|u_\pm(\mathsf{s})\rangle$ are eigenvectors of

$$\mathcal{H}'(\mathsf{s}) = \frac{s_x}{R_1}\sigma_3 + \frac{s_y}{R_2}\sigma_1 + \alpha\frac{s_x}{R_0} \tag{B.31}$$

with eigenvalues $\pm\varepsilon(\mathsf{s}) + \alpha s_x/R_0$, where $\varepsilon(\mathsf{s}) = \sqrt{(s_x/R_1)^2 + (s_y/R_2)^2}$. Because of P symmetry, $\sigma_3\mathcal{H}'(\mathsf{s})U\sigma_3^{-1} = \mathcal{H}'(\bar{\mathsf{s}})$, $\sigma_3|u_\pm(\bar{\mathsf{s}})\rangle$ are also eigenvectors of $\mathcal{H}'(\mathsf{s})$ with eigenvalues $\pm\varepsilon(\mathsf{s}) + \alpha s_x/R_0$, and therefore the off-diagonal matrix elements are zero, $\langle u_+(\mathsf{s})|\sigma_3|u_-(\bar{\mathsf{s}})\rangle = \langle u_-(\mathsf{s})|\sigma_3|u_+(\bar{\mathsf{s}})\rangle = 0$.

The diagonal elements, and hence, the transformation properties of $\chi_\pm(\mathsf{s})$ under parity, depend on a choice of eigen functions $\vec{u}_\pm(\mathsf{s})$. For $s_y \neq 0$, the following choice for the eigenvectors:

$$|u_\pm(\mathsf{s})\rangle = \frac{1}{\sqrt{2\varepsilon(\mathsf{s})\left[\varepsilon(\mathsf{s}) \pm s_x/R_1\right]}} \begin{bmatrix} s_x/R_1 \pm \varepsilon(\mathsf{s}) \\ s_y/R_2 \end{bmatrix} \tag{B.32}$$

leads to

$$\langle u_+(\mathsf{s})|\sigma_3|u_+(\bar{\mathsf{s}})\rangle = \langle u_-(\mathsf{s})|\sigma_3|u_-(\bar{\mathsf{s}})\rangle = 1. \tag{B.33}$$

Alternatively, a different gauge choice

$$|u_\pm(\mathsf{s})\rangle = \frac{1}{\sqrt{2\varepsilon(\mathsf{s})\left[\varepsilon(\mathsf{s}) \pm s_x/R_1\right]}} \begin{bmatrix} s_y/R_2 \\ -s_x/R_1 \pm \varepsilon(\mathsf{s}) \end{bmatrix} \tag{B.34}$$

leads to

$$\langle u_+(\mathsf{s})|\sigma_3|u_+(\bar{\mathsf{s}})\rangle = \langle u_-(\mathsf{s})|\sigma_3|u_-(\bar{\mathsf{s}})\rangle = -1. \tag{B.35}$$

In either choice, the result can be summarized as

$$\mathscr{P} \begin{bmatrix} \chi_+(\mathsf{s}) \\ \chi_-(\mathsf{s}) \end{bmatrix} \mathscr{P}^{-1} = \begin{bmatrix} \eta_+\chi_+(\bar{\mathsf{s}}) \\ \eta_-\chi_-(\bar{\mathsf{s}}) \end{bmatrix}, \quad s_y \neq 0, \tag{B.36}$$

where $\eta_\pm$ is an s-independent sign factor. Note that the condition $\mathscr{P}^2 = 1$ forces $\eta_\pm^2 = 1$. While $\eta_\pm$ depends on the choice of eigenfunctions, the final results (such as the evaluation of the partition functions) do not depend on such ambiguity.

On the parity-invariant line $s_y = 0$, which exists if $a_y \in \mathbb{Z}$, the Hamiltonian $\mathcal{H}'(s_x, s_y = 0) = \frac{s_x}{R_1}\sigma_3 + \alpha\frac{s_x}{R_0}$ has a "chiral decomposition":

$$|u_R(s_x)\rangle = \begin{bmatrix} e^{i\alpha_R} \\ 0 \end{bmatrix}, \quad |u_L(s_x)\rangle = \begin{bmatrix} 0 \\ e^{i\alpha_L} \end{bmatrix}, \quad \alpha_{R,L} \in \mathbb{R}, \tag{B.37}$$

which corresponds to the "chiral eigen basis" $\chi_{R,L}$. Since $\langle u_R(s_x)|\sigma_3|u_R(s_x)\rangle = -\langle u_L(s_x)|\sigma_3|u_L(s_x)\rangle = 1$

and $\langle u_R(s_x)|\sigma_3|u_L(s_x)\rangle = \langle u_L(s_x)|\sigma_3|u_R(s_x)\rangle = 0$ (independent of the choice of the phases $\alpha_{R/L}$), we have

$$\mathscr{P}\begin{bmatrix} \chi_R(s_x) \\ \chi_L(s_x) \end{bmatrix}\mathscr{P}^{-1} = \sigma_3 \begin{bmatrix} \chi_R(s_x) \\ \chi_L(s_x) \end{bmatrix}, \quad s_y = 0, \tag{B.38}$$

which does not depend on the normalizations of $|u_{R/L}(s_x)\rangle$. We observe, on the P-invariant line $s_y = 0$, parity acts like the "spin parity" $(-1)^{F_L}$, where F_L can be thought as the total number of $\chi_L(s_x)$ (at $s_y = 0$). Thus, we expect that the modular properties of this surface theory, as determined solely by the 2d massless modes $(s_x, s_y = 0)$, will be similar to the modular properties of the edge theory of (2+1)d topological superconductors protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetries [28].

P-twisted partition functions in the τ-direction First we evaluate the partition function twisted by P in the τ-direction, which can be written as

$$Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x}, a_y} = \mathrm{Tr}_{\mathscr{G}_f^{2a_x}, a_y}[\mathscr{P} e^{2\pi i(a_\tau - 1/2)F} e^{-2\pi R_0 H'}] = e^{-2\pi R_0 E_{\mathrm{GS}}} \prod_{s_x} W^{\mathrm{P}}_{[\mathsf{a}]}(s_x), \tag{B.39}$$

where $a_y = 0, 1/2$, $E_{\mathrm{GS}} = -\sum_{\mathsf{s}} \varepsilon(\mathsf{s})$, and $W^{\mathrm{P}}_{[\mathsf{a}]}(s_x)$ can be written in a pairwise fashion (with respect to P symmetry):

$$W^{\mathrm{P}}_{[\mathsf{a}]}(s_x) = X^{\mathrm{P}}_{[\mathsf{a}]}(s_x) \times Y^{\mathrm{P}+}_{[\mathsf{a}]}(s_x) \times Y^{\mathrm{P}-}_{[\mathsf{a}]}(s_x), \tag{B.40}$$

where

$$\begin{aligned} X^{\mathrm{P}}_{[\mathsf{a}]}(s_x) = \mathrm{Tr}_{a_x a_y} \mathscr{P} \exp\Big\{ & \\ & + 2\pi i\Big(a_\tau - \frac{1}{2}\Big)\Big[\chi_R^\dagger(s_x)\chi_R(s_x) + \chi_L^\dagger(s_x)\chi_L(s_x)\Big] \\ & - 2\pi s_x \frac{R_0}{R_1}\Big[\chi_R^\dagger(s_x)\chi_R(s_x) - \chi_L^\dagger(s_x)\chi_L(s_x)\Big] \\ & + 2\pi i \alpha s_x \Big[\chi_R^\dagger(s_x)\chi_R(s_x) + \chi_L^\dagger(s_x)\chi_L(s_x)\Big]\Big\} \end{aligned} \tag{B.41}$$

and

$$
\begin{aligned}
Y^{\mathrm{P}\pm}_{[\mathsf{a}]}(s_x) = \prod_{s_y>0} \mathrm{Tr}_{a_x a_y}\, \mathscr{P} \exp\Big\{ \\
& \pm 2\pi i\Big(a_\tau - \frac{1}{2}\Big) \left[\chi^\dagger_\pm(\mathsf{s})\chi_\pm(\mathsf{s}) + \chi^\dagger_\pm(\bar{\mathsf{s}})\chi_\pm(\bar{\mathsf{s}})\right] \\
& - 2\pi\varepsilon(\mathsf{s})\frac{R_0}{R_1} \left[\chi^\dagger_\pm(\mathsf{s})\chi_\pm(\mathsf{s}) + \chi^\dagger_\pm(\bar{\mathsf{s}})\chi_\pm(\bar{\mathsf{s}})\right] \\
& \pm 2\pi i \alpha s_x \left[\chi^\dagger_\pm(\mathsf{s})\chi_\pm(\mathsf{s}) + \chi^\dagger_\pm(\bar{\mathsf{s}})\chi_\pm(\bar{\mathsf{s}})\right] \Big\}.
\end{aligned}
\tag{B.42}
$$

Note that the 2d massless modes ($s_y = 0$) $X^{\mathrm{P}}_{[\mathsf{a}]}(s_x)$ would be present if $a_y = 0$. With such pairwise decomposition, the 2d massive part for fixed $s_y \neq 0$ in Eq. (B.39) is evaluated as

$$
e^{2\pi R_0 \sum_{s_x\in\mathbb{Z}+a_x} \varepsilon(\mathsf{s})} \prod_{s_x\in\mathbb{Z}+a_x} \left|1 - e^{-2\pi R_0\varepsilon(\mathsf{s}) + 2\pi i \alpha s_x + 2\pi i a_\tau}\right|^2 = \Theta_{[a_x, 2a_\tau]}\left(\tau_{2d}; r_{12} s_y\right), \tag{B.43}
$$

while the 2d massless part is evaluated as

$$
A^R_{[a_x,a_\tau]}(\tau_{2d}) A^L_{[a_x, a_\tau - \frac{1}{2}]}(\tau_{2d}). \tag{B.44}
$$

In summary,

$$
Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau}, \mathscr{G}_f^{2a_x}, a_y} = \begin{cases}
\text{const.} \times A^R_{[a_x,a_\tau]}(\tau_{2d}) A^L_{[a_x,a_\tau-\frac{1}{2}]}(\tau_{2d}) \prod_{s_y\in\mathbb{Z}^+} \Theta_{[a_x,2a_\tau]}\left(2\tau_{2d}; r_{12}s_y\right), \\
\qquad \text{for } a_y = 0 \text{ (PBC in the } y\text{-direction);} \\
\\
\text{const.} \times \displaystyle\prod_{s_y\in\mathbb{Z}^+-\frac{1}{2}} \Theta_{[a_x,2a_\tau]}\left(2\tau_{2d}; r_{12}s_y\right), \\
\qquad \text{for } a_y = 1/2 \text{ (APBC in the } y\text{-direction).}
\end{cases} \tag{B.45}
$$

Here the constant prefactors are related to the P eigenvalues of the ground states.

P-twisted partition functions in the x-direction Now let us consider the partition function twisted by P in the x-direction. We start with the twisted boundary conditions in the x- and y-directions:

$$
\begin{aligned}
\psi(x + 2\pi R_1, y) &= \big(\mathscr{P}\mathscr{G}_f^{2a_x}\big)\psi(x,y)\big(\mathscr{P}\mathscr{G}_f^{2a_x}\big)^{-1} = e^{2\pi i a_x}\sigma_3\psi(x,-y), \\
\psi(x, y + 2\pi R_2) &= \big(\mathscr{G}_f^{2a_x}\big)\psi(x,y)(\mathscr{G}_f^{2a_x}\big)^{-1} = e^{2\pi i a_y}\psi(x,y).
\end{aligned} \tag{B.46}
$$

With the above twisted boundary condition, the Fourier expansion of the fermion fields can be expressed as

$$\psi(\mathsf{r}) = \sum_{s_x\in\mathbb{Z}/2+a_x}\sum_{s_y\in\mathbb{Z}+a_y} e^{ix\frac{s_x}{R_1}+iy\frac{s_y}{R_2}}\left[\vec{u}_+(\mathsf{s})\chi_+(\mathsf{s})+\vec{u}_-(\mathsf{s})\chi_-(\mathsf{s})\right]+\{\text{2d massless modes}\}, \tag{B.47}$$

with

$$e^{2\pi i s_x}\chi_\pm(\mathsf{s}) = \eta_\pm e^{2\pi i a_x}\chi_\pm(\bar{\mathsf{s}}), \quad s_x\in\frac{\mathbb{Z}}{2}+a_x, \quad \eta_\pm^2=1, \tag{B.48}$$

where $\chi_\pm(\mathsf{s})$ are eigen basis of $\mathcal{H}'(\mathsf{s})$, $\vec{u}_\pm(\mathsf{s})$ are the corresponding eigenvectors [take the form of (B.32) or (B.34), up to normalization factors], and the term "2d massless modes" is present if $a_y\in\mathbb{Z}$. The 2d massless modes are given by the sum of the two terms

$$\begin{aligned}&\sum_{s_x^R\in\mathbb{Z}+a_x} e^{ix\frac{s_x^R}{R_1}}\vec{u}_R(s_x^R)\chi_R(s_x^R),\\ &\sum_{s_x^L\in\mathbb{Z}+a_x-\frac{1}{2}} e^{ix\frac{s_x^L}{R_1}}\vec{u}_L(s_x^L)\chi_L(s_x^L),\end{aligned} \tag{B.49}$$

where $\chi_{R,L}(\mathsf{s})$ are eigenbasis of $\mathcal{H}'(s_x, s_y=0)$ and $\vec{u}_{R,L}(\mathsf{s})$ are the corresponding eigenvectors in Eq. (B.37).

From the condition (B.48), which relates eigen modes with s and $\bar{\mathsf{s}}$, we only need to take "half" of the degree of freedoms, either modes with $s_y>0$ or with $s_y<0$, when we calculate the trace for the partition functions. The result does not depend on which region for s_y we choose. From the above discussion, the 2d massive part for fixed $s_y\neq 0$ in the trace $\mathrm{Tr}_{\mathscr{P}\mathscr{G}_f^{2a_x},a_y}[\mathscr{G}_f^{2(a_\tau-1/2)}e^{-2\pi R_0 H'}]$ is evaluated as

$$e^{2\pi R_0\sum_{s_x\in\mathbb{Z}/2+a_x}\varepsilon(\mathsf{s})}s\prod_{s_x\in\mathbb{Z}/2+a_x}\left|1-e^{-2\pi R_0\varepsilon(\mathsf{s})+2\pi i\alpha s_x+2\pi i a_\tau}\right|^2 = \Theta_{[2a_x,a_\tau]}\left(\tau_{2d}/2; r_{12}s_y\right), \tag{B.50}$$

while the 2d massless part (if present) is evaluated as

$$A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x-\frac{1}{2},a_\tau]}(\tau_{2d}). \tag{B.51}$$

In summary,

$$
Z_{\mathscr{G}_f^{2a_\tau},\mathscr{P}\mathscr{G}_f^{2a_x},a_y} = \begin{cases} \text{const.} \times A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x-\frac{1}{2},a_\tau]}(\tau_{2d}) \prod_{s_y\in\mathbb{Z}^+} \Theta_{[2a_x,a_\tau]}\left(\tau_{2d}/2;2r_{12}s_y\right), \\ \qquad \text{for } a_y=0 \text{ (PBC in the } y\text{-direction);} \\ \\ \text{const.} \times \prod_{s_y\in\mathbb{Z}^+-\frac{1}{2}} \Theta_{[2a_x,a_\tau]}\left(\tau_{2d}/2;2r_{12}s_y\right), \\ \qquad \text{for } a_y=1/2 \text{ (APBC in the } y\text{-direction).} \end{cases} \tag{B.52}
$$

P-twisted partition functions in the τ- and x-directions Finally, we calculate the partition function twisted by P both in the τ- and x-directions, $Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau},\mathscr{P}\mathscr{G}_f^{2a_x},a_y}$. Using the result from the last section, we now just need to include the additional insertion of the parity operator inside the trace. This can be done by observing that

$$
\mathscr{P}\chi_\pm(\mathsf{s})\mathscr{P}^{-1} = \eta_\pm\chi_\pm(\bar{\mathsf{s}}) = e^{2\pi i s_x}e^{-2\pi i a_x}\chi_\pm(\mathsf{s}) \tag{B.53}
$$

for the massive modes ($s_y \neq 0$) and

$$
\mathscr{P}\begin{bmatrix} \chi_R(s_x) \\ \chi_L(s_x) \end{bmatrix}\mathscr{P}^{-1} = \sigma_3 \begin{bmatrix} \chi_R(s_x) \\ \chi_L(s_x) \end{bmatrix}
$$

for the massless modes (where $s_y = 0$ as usual). Then, the 2d massive part for fixed $s_y \neq 0$ in the trace is evaluated as

$$
e^{2\pi R_0 \sum_{s_x\in\mathbb{Z}/2+a_x}\varepsilon(\mathsf{s})} \prod_{s_x\in\mathbb{Z}/2+a_x} \left|1-e^{-2\pi R_0\varepsilon(\mathsf{s})+2\pi i(\alpha+1)s_x+2\pi i(a_\tau-a_x)}\right|^2 = \Theta_{[2a_x,a_\tau-a_x]}\left(\tau_{2d}/2+1/2;2r_{12}s_y\right), \tag{B.54}
$$

while the 2d massless part is evaluated as

$$
A^R_{[a_x,a_\tau]}(\tau_{2d})A^L_{[a_x-\frac{1}{2},a_\tau-\frac{1}{2}]}(\tau_{2d}). \tag{B.55}
$$

In summary,

$$
Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau},\mathscr{P}\mathscr{G}_f^{2a_x},a_y} = \begin{cases} \text{const.} \times A^R_{[a_x,a_\tau]}(\tau_{2d}) A^L_{[a_x-\frac{1}{2},a_\tau-\frac{1}{2}]}(\tau_{2d}) \prod_{s_y\in\mathbb{Z}^+} \Theta_{[2a_x,a_\tau-a_x]}\left(\tau_{2d}/2+1/2; 2r_{12}s_y\right), \\ \qquad \text{for } a_y = 0 \text{ (PBC in the } y\text{-direction);} \\ \\ \text{const.} \times \prod_{s_y\in\mathbb{Z}^+-\frac{1}{2}} \Theta_{[2a_x,a_\tau-a_x]}\left(\tau_{2d}/2+1/2; 2r_{12}s_y\right), \\ \qquad \text{for } a_y = 1/2 \text{ (APBC in the } y\text{-direction).} \end{cases} \tag{B.56}
$$

B.5 Massive modes $\Theta^{i-iv}_{[a_x,a_\tau]}(\tau_{2d}; r_{12})$ under $\mathrm{SL}(2,\mathbb{Z})$ transformations

In this Appendix, we discuss how the (products of) massive modes $\Theta^{i-iv}_{[a_x,a_\tau]}(\tau_{2d}; r_{12})$, defined in Eq. (3.64), transform under $\mathrm{SL}(2,\mathbb{Z})$ generated by U_1' and U_2. This can be deduced from the modular properties (3.5) of the massive theta functions with modular parameters τ_{2d}, $2\tau_{2d}$, $\tau_{2d}/2$, and $\tau_{2d}/2+1/2$ (we denote the mass parameter $m = r_{12}s_y$ in the following equations):

(i) For $\Theta_{[a_x,a_\tau]}(\tau_{2d}; m)$:

$$
\begin{aligned}
&\Theta_{[a_x,a_\tau]}(\tau_{2d}; m) \xrightarrow{U_1'} \Theta_{[a_x,a_\tau]}\left(-1/\tau_{2d}; m|\tau_{2d}|\right) = \Theta_{[-a_\tau,a_x]}\left(\tau_{2d}; m\right), \\
&\Theta_{[a_x,a_\tau]}(\tau_{2d}; m) \xrightarrow{U_2^{-1}} \Theta_{[a_x,a_\tau]}(\tau_{2d}+1; m) = \Theta_{[a_x,a_x+a_\tau]}\left(\tau_{2d}; m\right);
\end{aligned} \tag{B.57}
$$

(ii) For $\Theta_{[a_x,a_\tau]}(2\tau_{2d}; m)$:

$$
\begin{aligned}
&\Theta_{[a_x,a_\tau]}(2\tau_{2d}; m) \xrightarrow{U_1'} \Theta_{[a_x,a_\tau]}\left(-2/\tau_{2d}; m|\tau_{2d}|\right) = \Theta_{[-a_\tau,a_x]}\left(\tau_{2d}/2; 2m\right), \\
&\Theta_{[a_x,a_\tau]}(2\tau_{2d}; m) \xrightarrow{U_2^{-1}} \Theta_{[a_x,a_\tau]}(2\tau_{2d}+2; m) = \Theta_{[a_x,2a_x+a_\tau]}\left(2\tau_{2d}; m\right);
\end{aligned} \tag{B.58}
$$

(iii) For $\Theta_{[a_x,a_\tau]}(\tau_{2d}/2; 2m)$:

$$
\begin{aligned}
&\Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2; 2m\right) \xrightarrow{U_1'} \Theta_{[a_x,a_\tau]}\left(-1/2\tau_{2d}; 2m|\tau_{2d}|\right) = \Theta_{[-a_\tau,a_x]}\left(2\tau_{2d}; m\right), \\
&\Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2; 2m\right) \xrightarrow{U_2^{-1}} \Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2+1/2; 2m\right);
\end{aligned} \tag{B.59}
$$

(iv) For $\Theta_{[a_x,a_\tau]}(\tau_{2d}/2+1/2;2m)$:

$$\begin{aligned}
&\Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2+1/2;2m\right) \xrightarrow{U_1'} \Theta_{[a_x,a_\tau]}\left(-1/2\tau_{2d}+1/2;2m|\tau_{2d}|\right) = \Theta_{[-a_x-2a_\tau,a_x+a_\tau]}\left(\tau_{2d}/2+1/2;2m\right), \\
&\Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2+1/2;2m\right) \xrightarrow{U_2^{-1}} \Theta_{[a_x,a_\tau]}\left(\tau_{2d}/2+1;2m\right) = \Theta_{[a_x,a_x+a_\tau]}\left(\tau_{2d}/2;2m\right).
\end{aligned} \tag{B.60}$$

Therefore,

$$\begin{aligned}
&\Theta^{i}_{[a_x,a_\tau]} \xrightarrow{U_1'} \Theta^{i}_{[-a_\tau,a_x]}, \quad \Theta^{i}_{[a_x,a_\tau]} \xrightarrow{U_2^{-1}} \Theta^{i}_{[a_x,a_x+a_\tau]}, \\
&\Theta^{ii}_{[a_x,a_\tau]} \xrightarrow{U_1'} \Theta^{iii}_{[-a_\tau,a_x]}, \quad \Theta^{ii}_{[a_x,a_\tau]} \xrightarrow{U_2^{-1}} \Theta^{ii}_{[a_x,2a_x+a_\tau]}, \\
&\Theta^{iii}_{[a_x,a_\tau]} \xrightarrow{U_1'} \Theta^{ii}_{[-a_\tau,a_x]}, \quad \Theta^{iii}_{[a_x,a_\tau]} \xrightarrow{U_2^{-1}} \Theta^{iv}_{[a_x,a_\tau]}, \\
&\Theta^{iv}_{[a_x,a_\tau]} \xrightarrow{U_1'} \Theta^{iv}_{[-a_x-2a_\tau,a_x+a_\tau]}, \quad \Theta^{iv}_{[a_x,a_\tau]} \xrightarrow{U_2^{-1}} \Theta^{iii}_{[a_x,a_x+a_\tau]}.
\end{aligned} \tag{B.61}$$

B.6 $\mathrm{SL}(2,\mathbb{Z})$ invariance of the total partition function for $a_y = 1/2$

The parity-twisted partition functions for $a_y = 1/2$, as computed in Appendix B.4, are summarized as follows:

$$\begin{aligned}
&Z_{\mathscr{G}_f^{2a_\tau},\mathscr{G}_f^{2a_x},a_y=\frac{1}{2}} = \tilde{\Theta}^{i}_{[a_x,a_\tau]}(\tau_{2d};r_{12}), \\
&Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau},\mathscr{G}_f^{2a_x},a_y=\frac{1}{2}} = \text{const.} \times \tilde{\Theta}^{ii}_{[a_x,2a_\tau]}(\tau_{2d};r_{12}), \\
&Z_{\mathscr{G}_f^{2a_\tau},\mathscr{P}\mathscr{G}_f^{2a_x},a_y=\frac{1}{2}} = \text{const.} \times \tilde{\Theta}^{iii}_{[2a_x,a_\tau]}(\tau_{2d};r_{12}), \\
&Z_{\mathscr{P}\mathscr{G}_f^{2a_\tau},\mathscr{P}\mathscr{G}_f^{2a_x},a_y=\frac{1}{2}} = \text{const.} \times \tilde{\Theta}^{iv}_{[2a_x,a_\tau-a_x]}(\tau_{2d};r_{12}),
\end{aligned} \tag{B.62}$$

where we have introduced $\tilde{\Theta}^{i-iv}_{[a_x,a_\tau]}(\tau_{2d};r_{12})$ as:

$$\begin{aligned}
&\tilde{\Theta}^{i}_{[a_x,a_\tau]} = \prod_{s_y\in\mathbb{Z}^+-1/2} \left[\Theta_{[a_x,a_\tau]}\left(\tau_{2d};r_{12}s_y\right)\right]^2, \\
&\tilde{\Theta}^{ii}_{[a_x,a_\tau]} = \prod_{s_y\in\mathbb{Z}^+-1/2} \Theta_{[a_x,a_\tau]}(2\tau_{2d};r_{12}s_y), \\
&\tilde{\Theta}^{iii}_{[a_x,a_\tau]} = \prod_{s_y\in\mathbb{Z}^+-1/2} \Theta_{[a_x,a_\tau]}(\tau_{2d}/2;2r_{12}s_y), \\
&\tilde{\Theta}^{iv}_{[a_x,a_\tau]} = \prod_{s_y\in\mathbb{Z}^+-1/2} \Theta_{[a_x,a_\tau]}(\tau_{2d}/2+1/2;2r_{12}s_y).
\end{aligned} \tag{B.63}$$

The constant prefactors are again related to the P eigenvalues of the ground states, which can be absorbed to the (redefined) weights as we consider the partition sum.

The total partition function is then given by

$$\begin{aligned}\mathcal{Z}^{tot}_{[a_y=\frac{1}{2}]}(g_{\mathrm{P}}) &= \epsilon_1\tilde{\Theta}^{i}_{[0,0]} + \epsilon_2\tilde{\Theta}^{i}_{[0,\frac{1}{2}]} + \epsilon_3\tilde{\Theta}^{i}_{[\frac{1}{2},0]} + \epsilon_4\tilde{\Theta}^{i}_{[\frac{1}{2},\frac{1}{2}]} \\ &\quad + 2\Big(\epsilon_5\tilde{\Theta}^{ii}_{[0,0]} + \epsilon_6\tilde{\Theta}^{ii}_{[\frac{1}{2},0]} + \epsilon_7\tilde{\Theta}^{iii}_{[0,0]} + \epsilon_8\tilde{\Theta}^{iii}_{[0,\frac{1}{2}]} + \epsilon_9\tilde{\Theta}^{iv}_{[0,0]} + \epsilon_{10}\tilde{\Theta}^{iv}_{[0,\frac{1}{2}]}\Big).\end{aligned} \tag{B.64}$$

From the modular properties of Θ (and thus of $\tilde{\Theta}^{i-iv}$) discussed in Appendix B.5, we can see that $\mathcal{Z}^{tot}_{[a_y=\frac{1}{2}]}(g_{\mathrm{P}})$ can be made $\mathrm{SL}(2,\mathbb{Z})$ (generated by U_1' and U_2) invariant for any number of Dirac fermion flavors, N, if we choose $\epsilon_i = 1$ for all i (more precisely, we just need $\epsilon_2 = \epsilon_3 = \epsilon_4$ and $\epsilon_5 = ... = \epsilon_{10}$).

Appendix C

C.1 Details of the derivation of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ and $\Gamma^5_{\mathrm{Spin}}(B\mathbb{Z}_n)$

In this Appendix, we give the details of the derivation of Eq. (4.15) in Sec. C.1.1 and of Eqs. (4.22) and (4.24) in Sec. C.1.2.

C.1.1 Derivation of Eq. (4.15)

We first show that $S(n)$ can be expressed in terms of the representation theory of $\mathbb{Z}_n$. Here we follow the idea in [131] for the case $n = 2^v$ and generalize their result to any prime power $n = p^v$. Specifically, for each $n = p^v$, we have

$$S(n) \cong I(n)/\{I(n) \cap RU_0(\mathbb{Z}_n)^4\}, \tag{C.1}$$

where

$$I(n) := \{R = \oplus_i \rho_{s_i}(\lambda) \in RU_0(\mathbb{Z}_n) : R(\bar{\lambda}) = -R(\lambda),\ \lambda \in \mathbb{Z}_n\} \tag{C.2}$$

and $RU_0(\mathbb{Z}_n)$ is the argumentation ideal of representations of $\mathbb{Z}_n$ with virtual dimension 0, that is,

$$RU_0(\mathbb{Z}_n) = (\rho_1 - \rho_0) \cdot RU(\mathbb{Z}_n). \tag{C.3}$$

Here $\rho_s = \lambda^s$ is a one-dimensional representation of $\mathbb{Z}_n$. Eq. (C.1) can be proved by relating the mod $\mathbb{Z}$ eta-invariant on an element of $S(n)$ to the mod $\mathbb{Z}$ eta-invariant on a seven-dimensional lens space $L(n; 1, a, 1, -1)$

and by constructing an explicit isomorphism between $S(n)$ and $I(n)/\{I(n) \cap RU_0(\mathbb{Z}_n)^4\}$ for each $n = p^v$ through this relation. Here

$$L(n; a_1, a_2, a_3, a_4) := S^7/\tau(a_1, a_2, a_3, a_4), \tag{C.4}$$

where $\tau(a_1, a_2, a_3, a_4) := \rho_{a_1} \oplus \rho_{a_2} \oplus \rho_{a_3} \oplus \rho_{a_4}$ is a representation of $\mathbb{Z}_n$ in $\mathrm{U}(4)$ and its action (by multiplication by λ^{a_i} on the i-th summand) on the associated unit sphere bundle is fixed-point free. By construction, the lens spaces inherit natural spin structures and $\mathbb{Z}_n$ structures (similar to the case of lens space bundles $X(n; a_1, a_2)$).

The eta-invariant on $L(n; a_1, a_2, a_3, a_4)$ associated with a representation $R = \oplus_i \rho_{s_i} \in RU(\mathbb{Z}_n)$ can be computed by the following formula [132, 131]

$$\eta(L(n; a_1, a_2, a_3, a_4), R) = \frac{1}{n} \sum_{\lambda \in \mathbb{Z}_n, \lambda \neq 1} \mathrm{Tr}(R(\lambda)) \frac{\lambda^{\frac{1}{2}(a_1+a_2+a_3+a_4)}}{(1-\lambda^{a_1})(1-\lambda^{a_2})(1-\lambda^{a_3})(1-\lambda^{a_4})}. \tag{C.5}$$

Let $\gamma := \rho_1 - \rho_{-1}$ and $\xi := \rho_{-1}(\rho_0 - \rho_1)^2$. We define an additive map $\sigma_n : S(n) \to RU_0(\mathbb{Z}_n)$ for each $n = p^v$ via the following relations on generators of $S(n)$:

$$\begin{cases} \sigma_n([X(n;1,1)]_\eta) = \gamma, & \text{if } n = 2, 3, \\ \sigma_n([X(n;1,3)]_\eta) = \gamma, \quad \sigma_n([X(n;1,1)]_\eta - 3[X(n;1,3)]_\eta) = \gamma\xi, & \text{if } n = 2^v > 2, \\ \sigma_n([X(n;1,5)]_\eta) = \gamma, \quad \sigma_n([X(n;1,1)]_\eta - 5[X(n;1,5)]_\eta) = 5\gamma\xi + \gamma\xi^2, & \text{if } n = 3^v > 3, \\ \sigma_n([X(n;1,3)]_\eta) = \gamma, \quad \sigma_n([X(n;1,1)]_\eta - 3[X(n;1,3)]_\eta) = \gamma\xi, & \text{if } n = p^v,\ p > 3. \end{cases} \tag{C.6}$$

Then, for any $[X]_\eta \in S(n)$ (given $n = p^v$) and for any $R \in RU(\mathbb{Z}_n)$, we have

$$\eta([X]_\eta, R) = \eta([L(n; 1, k_n, 1, -1)]_\eta, \sigma_n([X]_\eta)R) \mod \mathbb{Z}, \tag{C.7}$$

where $k_n = 1$, if $n = 2, 3$; $k_n = 3$, if $n = p^v > 2,\ p \neq 3$; $k_n = 5$, if $n = 3^v > 3$. Eq. (C.7) can be checked directly using the formulas of the eta-invariants on the lens space bundles and on the lens spaces, that is, Eqs. (4.13) and (C.5). Here we skip the computation details.

For each $n = p^v$, if $[X]_\eta = 0$ in $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$, we will have $\eta([X]_\eta, R) = 0 \mod \mathbb{Z}$ for all $R \in RU(\mathbb{Z}_n)$, by the definition of $\Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$. From (C.7), we then have $\eta([L(n; 1, k_n, 1, -1)]_\eta, \sigma_n(0)R) = 0 \mod \mathbb{Z}$ for all $R \in RU(\mathbb{Z}_n)$ (and thus all $R \in RU_0(\mathbb{Z}_n)$). This implies $\sigma_n(0) \in RU_0(\mathbb{Z}_n)^4$ for $n = 2^v$ [131] and for $n = p^v$ with p being an odd prime [130]. Therefore, we can regard σ_n as a well-defined map from $RU_0(\mathbb{Z}_n)$ to $RU_0(\mathbb{Z}_n)/RU_0(\mathbb{Z}_n)^4$.

Now we show that the map σ_n is an isomorphsim from $S(n)$ to $I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}$ for each $n=p^v$. We argue this as follows. If $\sigma_n([X]_\eta)=0$, from (C.7) we have $\eta([X]_\eta,R)=0 \mod \mathbb{Z}_n$ for all $R\in RU(\mathbb{Z}_n)$. Again, by the definition of $\Gamma_5^{\rm Spin}(B\mathbb{Z}_n)$, $[X]_\eta=0$ in $\Gamma_5^{\rm Spin}(B\mathbb{Z}_n)$ and, of course, in $S(n)$. So σ_n is injective. On the other hand, the set $I(n)$ defined in (C.2) is generated by $\rho_s-\rho_{-s}$ for $s=0,...,n-1$ or equivalently by $\gamma\xi^j$ for $j\geq 0$. It is obvious that σ_n for $n=2,3,4$ is surjective, since $I(2)=0$ and $I(3)=I(4)=\gamma\cdot\mathbb{Z}$. For $n=p^v>4$, $I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}$ is generated by γ and $\gamma\xi$ only (as $\gamma\xi^j$ for $j\geq 2$ is 0 modulo $RU_0(\mathbb{Z}_n)^4$), so σ_n is surjective from $S(n)$ to $I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}$. This completes the proof of (C.1).

The representation theory $I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}$ of $\mathbb{Z}_n$ for each $n=p^v$ can be expressed in terms of (direct sums of) cyclic groups. The case of $n=2$ is trivial as $I(2)=0$. For $n=p^v>2$, we express $I(n)=\oplus_{j\geq 0}\gamma\xi^j\cdot\mathbb{Z}=\oplus_{s=0}^{n-1}(\rho_s-\rho_{-s})\cdot\mathbb{Z}$ modulo the condition $(\rho_s-\rho_0)^4=1$ (as well as $\rho_s^n=1$), writing $x=\rho_1-\rho_0\in RU_0(\mathbb{Z}_n)$, as

$$\begin{aligned}
&I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}\\
&=\{\gamma\cdot\mathbb{Z}\oplus\gamma\xi\cdot\mathbb{Z}\}/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}\\
&=\{(\rho_1-\rho_{-1})\cdot\mathbb{Z}\oplus(\rho_2-\rho_{-2})\cdot\mathbb{Z}\}/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}\\
&=\{\{(1+x)-(1+x)^{n-1}\}\cdot\mathbb{Z}\oplus\{(1+x)^2-(1+x)^{n-2}\}\cdot\mathbb{Z}\}/\{\{(1+x)^n-1\}\cdot\mathbb{Z}[x]+x^4\cdot\mathbb{Z}[x]\}\\
&=\left\{\left\{(n-2)x+\tbinom{n-1}{2}x^2+\tbinom{n-1}{3}x^3\right\}\cdot\mathbb{Z}\oplus\left\{(n-4)x+\left[\tbinom{n-2}{2}-1\right]x^2+\tbinom{n-2}{3}x^3\right\}\cdot\mathbb{Z}\right\}\\
&\quad/\left\{\left\{nx+\tbinom{n}{2}x^2+\tbinom{n}{3}x^3\right\}\cdot\mathbb{Z}[x]+x^4\cdot\mathbb{Z}[x]\right\}\\
&=\left\{\left\{(n-2)x+\tbinom{n-1}{2}x^2+\tbinom{n-1}{3}x^3\right\}\cdot\mathbb{Z}\oplus\left\{(n-4)x+\left[\tbinom{n-2}{2}-1\right]x^2+\tbinom{n-2}{3}x^3\right\}\cdot\mathbb{Z}\right\}\\
&\quad/\left\{\left\{nx+\tbinom{n}{2}x^2+\tbinom{n}{3}x^3\right\}\cdot\mathbb{Z}+nx^3\cdot\mathbb{Z}\right\}.
\end{aligned}\tag{C.8}$$

Then we want to find the minimal positive integers M_n and N_n (for each $n=p^v>2$) that satisfy

$$\begin{aligned}
&A_n\cdot\left\{nx+\tbinom{n}{2}x^2+\tbinom{n}{3}x^3\right\}+B_n\cdot nx^3=M_n\cdot\left\{(n-2)x+\tbinom{n-1}{2}x^2+\tbinom{n-1}{3}x^3\right\},\\
&C_n\cdot\left\{nx+\tbinom{n}{2}x^2+\tbinom{n}{3}x^3\right\}+D_n\cdot nx^3=N_n\cdot\left\{(n-4)x+\left[\tbinom{n-2}{2}-1\right]x^2+\tbinom{n-2}{3}x^3\right\},
\end{aligned}\tag{C.9}$$

where A_n, B_n, C_n, and D_n (which are not unique) are integers. For $n=3,4$, there is actually one generator for $I(n)/\{I(n)\cap RU_0(\mathbb{Z}_n)^4\}$, that is, $\rho_1-\rho_{-1}$, and we found that $M_3=9$ and $M_4=4$. For $n=p^v>4$, there are two linearly independent generators; analyzing in three classes, $p=2,3$, or an odd prime other than 3, we found

$$\begin{cases} M_n = n, \; N_n = n/4, & \text{if } n = 2^v > 4, \\ M_n = 3n, \; N_n = n/3, & \text{if } n = 3^v > 3, \\ M_n = n, \; N_n = n, & \text{if } n = p^v, \; p > 3. \end{cases} \tag{C.10}$$

Therefore, we have

$$I(n)/\{I(n) \cap RU_0(\mathbb{Z}_n)^4\} \cong \begin{cases} 0, & \text{if } n = 2, \\ \mathbb{Z}_n \oplus \mathbb{Z}_{n/4}, & \text{if } n = 2^v > 2, \\ \mathbb{Z}_{3n} \oplus \mathbb{Z}_{n/3}, & \text{if } n = 3^v, \\ \mathbb{Z}_n \oplus \mathbb{Z}_n, & \text{if } n = p^v, \; p > 3. \end{cases} \tag{C.11}$$

Together with (C.1), we thus prove Eq. (4.15).

C.1.2 Derivation of Eqs. (4.22) and (4.24)

Denote the lens space bundle $X_n := X(n; 1, 1)$. From the formulas of the eta-invariants on the five-dimensional lens space bundles (4.13) and on the seven-dimensional lens spaces (C.5), we have

$$\eta([X_n]_\eta, R) = \eta([L(n; 1, 1, 1, -1)]_\eta, (\rho_1 - \rho_{-1})R) \mod \mathbb{Z}, \quad \forall R \in RU(\mathbb{Z}), \quad \forall n = p^v. \tag{C.12}$$

On the other hand, the eta-invariant on $L(n; 1, 1, 1, -1)$ with a one-dimensional representation $\rho_s \in RU(\mathbb{Z})$ can be expressed, in terms of $\hat{A}_k(t; \vec{x})$ defined in (4.20), as [134]

$$\eta([L(n; 1, 1, 1, -1)]_\eta, \rho_s) = \frac{1}{n} \cdot \hat{A}_4(s + n/2; 1, 1, 1, -1) \mod \mathbb{Z}. \tag{C.13}$$

Thus we have

$$\begin{aligned} \eta([X_n]_\eta, \rho_s) &= \eta([L(n; 1, 1, 1, -1)]_\eta, \rho_{s+1} - \rho_{s-1}) \mod \mathbb{Z} \\ &= \frac{1}{n} \Big(\hat{A}_4(s + 1 + n/2; 1, 1, 1, -1) - \hat{A}_4(s - 1 + n/2; 1, 1, 1, -1) \Big) \mod \mathbb{Z} \\ &= \frac{1}{n} \left\{ \frac{1}{24} \left[(s + 1 + n/2)^4 - (s - 1 + n/2)^4 \right] \mod \mathbb{Z} \right. \\ &\quad \left. - \frac{1}{48} \left[(s + 1 + n/2)^2 - (s - 1 + n/2)^2 \right] \left(n^2 + 1^2 + 1^2 + 1^2 + (-1)^2 \right) \right\} \mod \mathbb{Z} \\ &= \frac{1}{6n} \left(2s^3 + 3ns^2 + n^2 s \right) \mod \mathbb{Z}, \end{aligned} \tag{C.14}$$

which gives Eq. (4.22) for a generic representation $R = \oplus_i \rho_{s_i}$.

Furthermore, we would like to show there exists an element $[Y_n]_\eta \in S(n) = \Gamma_5^{\mathrm{Spin}}(B\mathbb{Z}_n)$ such that

$$\eta([Y_n]_\eta, R) = \eta([L(n;1,1,1,-1)]_\eta, (\rho_2 - \rho_{-2})R) \mod \mathbb{Z}, \quad \forall R \in RU(\mathbb{Z}), \quad \forall n = p^v. \tag{C.15}$$

This is based on the fact that $S(n)$ is isomorphic to $I(n)/\{I(n) \cap RU_0(\mathbb{Z}_n)^4$ that is generated by $\rho_1 - \rho_{-1}$ and $\rho_2 - \rho_{-2}$, as shown in (C.1.1). More specifically, let τ_n be an isomorphism between $S(n)$ and $I(n)/\{I(n) \cap RU_0(\mathbb{Z}_n)^4$ such that $\tau_n([X_n]_\eta) = \rho_1 - \rho_{-1}$ (note that τ_n is different from σ_n defined in (C.6)). Then, the relation (C.12) can be extended to

$$\eta([X]_\eta, R) = \eta([L(n;1,1,1,-1)]_\eta, \tau_n([X]_\eta)R) \mod \mathbb{Z}, \quad [X]_\eta \in S(n), \quad \forall R \in RU(\mathbb{Z}), \quad \forall n = p^v. \tag{C.16}$$

Since there exists an element of $S(n)$, say, $[Y_n]_\eta$, such that $\tau_n([Y_n]_\eta) = \rho_2 - \rho_{-2}$, the above relation immediately implies (C.15). Therefore,

$$\begin{aligned}
\eta([Y_n]_\eta, \rho_s) &= \eta([L(n;1,1,1,-1)]_\eta, \rho_{s+2} - \rho_{s-2}) \mod \mathbb{Z} \\
&= \frac{1}{n}\Big(\hat{A}_4(s+2+n/2;1,1,1,-1) - \hat{A}_4(s-2+n/2;1,1,1,-1)\Big) \mod \mathbb{Z} \\
&= \frac{1}{3n}\left[2s^3 + (n^2+6)s\right] \mod \mathbb{Z},
\end{aligned} \tag{C.17}$$

which gives Eq. (4.24) for a generic representation $R = \oplus_i \rho_{s_i}$.

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